J. DIFFERENTIAL GEOMETRY
60 (2002) 103-153

AN EMBEDDED MINIMAL SURFACE WITH NO SYMMETRIES

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Abstract

We construct embedded minimal surfaces of finite total curvature in euclidean space by gluing catenoids and planes. We use Weierstrass Representation and we solve the Period Problem using the Implicit Function Theorem. As a corollary, we obtain the existence of minimal surfaces with no symmetries.

1. Introduction

This paper describes a new method to construct minimal surfaces in euclidean space which are complete, properly embedded and of finite total curvature.

The first such example, besides the plane and catenoid, was constructed by C. Costa in 1984, and generalised by D. Hoffman and W. Meeks. These surfaces, which are known as the Costa Hoffman Meeks family, are constructed using Weierstrass Representation. Their symmetries are used in an essential way to reduce the number of periods to 2.

N. Kapouleas [6] was the first one to construct large families of embedded minimal surfaces of finite total curvature. His examples may be seen as desingularisation of a family of catenoids with the same axis. They have (unestimably) high genus and order of dihedral symmetry. The construction uses partial differential equations and a fixed point theorem.

Received March 4, 2002.

Recently M. Weber and M. Wolf [11] have constructed minimal surfaces using Weierstrass Representation and Teichmuller Spaces techniques to solve the Period Problem. Their examples have eight symmetries and low genus (in the sense that the example with N ends has genus $N-2$, the smallest possible value according to the Hoffman Meeks conjecture). They can handle an arbitrary large number of periods, but it seems hard to prove that their examples are embedded.

In this paper we construct minimal surfaces with no restriction a priori on the genus and the number of symmetries. The surfaces we obtain may be seen as N parallel planes with small catenoidal necks between them (the planes are perturbed to have logarithmic growth at infinity).

As in the work of N. Kapouleas, they are obtained by perturbation of a singular object. The method however, is completely different, and is inspired from the proof of the uniqueness of the Riemann Example by W. Meeks, J. Perez and A. Ros [8]. We define quite explicitly the Weierstrass data of our surfaces and we solve the Period Problem using the Implicit Function Theorem at a singular point.

As an application of our main theorem, we obtain the existence of embedded minimal surfaces in $\mathbb{R}^3$ which have no nontrivial symmetries (by a trivial symmetry we mean the identity). The question of whether such a surface might exist was raised in [5], Section 5.2.

1.1 The Costa Hoffman Meeks family

Before stating our general theorem, we describe in some detail the Costa Hoffman Meeks family, which we will recover as a particular case. This is the only case where pictures are available, and I hope Figure 1 will help to visualise what the examples we construct look like.

The Costa Hoffman Meeks family depends on two parameters: an integer $m \geq 2$ and a real $x \geq 1$. Each surface has genus $m-1$, three ends and m vertical planes of symmetry. x is a modulus for the underlying Riemann surface. $m = 2$, $x = 1$ yields the original Costa surface. See [5] for the details of the construction of these examples.

What is relevant for us is the behaviour of the family as $x \to \infty$. What we observe in pictures is that for large values of x, the surface looks like three "planes", with one "neck" between the first and second planes (we say this is the neck at *level* one) and m necks between the second and third planes (we say these are the necks at level 2). After suitable scaling, the three "planes" converge when $x \to \infty$ to the horizontal plane $x_3 = 0$, and the necks collapse to $m+1$ distinct points in this plane, which we call their limit *position*. From the symmetries of the surface, it is clear that the top necks converge to the vertices of a regular m-gon and the bottom neck converges to the center of this polygon. In other words, the limit (after suitable scaling) of the Costa Hoffman Meeks family when $x \to \infty$ is a 3-sheeted plane with $m+1$ singular points placed at the vertices and center of a regular m-gon.

Under a larger scale, each neck converges, after suitable translation, to a catenoid (this follows, for instance, from compactness results [9]). We call the radius of the neck of this catenoid the limit *size* of the neck. What is relevant is the ratio between the limit sizes of the necks.

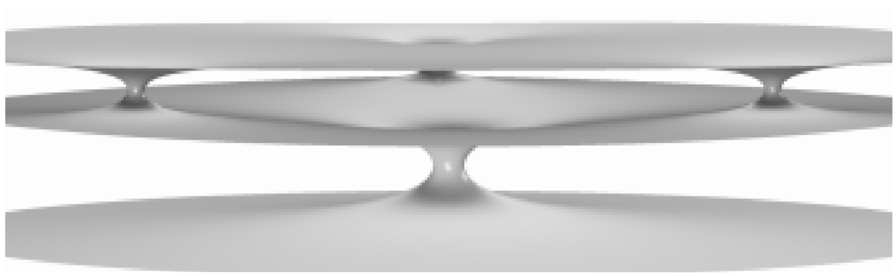

Figure 1: The genus 2 Costa Hoffman Meeks surface for large value of the parameter x. Computer image by J. Hoffman.

1.2 Configurations

We generalise this situation by allowing more planes and necks. The input data for our construction is the level, position and size of the necks. We call this a *configuration*. The configuration describes the asymptotic behaviour of the family of minimal surfaces we want to construct. More formally a configuration is the following data:

- an integer $N \geq 2$ (number of ends),
- a finite set I used to label the necks,
- for each $i \in I$, an integer ℓ_i, $1 \leq \ell_i \leq N-1$ (level of the i^{th} neck — here the word level has a combinatorial meaning, as explained in the Costa Hoffman Meeks case),
- for each $i \in I$, a complex number p_i (position of the i^{th} neck, identifying $\mathbb{R}^2$ with $\mathbb{C}$),

- for each k, $1 \leq k \leq N-1$, a real number $c_k > 0$ (size of the necks at level k).

We use the following notation: $I_k = \{i \in I \mid \ell_i = k\}$, $n_k = \#I_k$ is the number of necks at level k, $n = \#I$ is the number of necks. Note that $I_0 = I_N = \emptyset$ and $n_0 = n_N = 0$.

1.3 Forces

The configuration must satisfy a balancing condition which we now explain. Given $i \in I_k$, we define the force F_i by

$$F_i = \sum_{j \in I_k,\, j \neq i} \frac{2c_k^2}{p_i - p_j} - \sum_{j \in I_{k+1}} \frac{c_k c_{k+1}}{p_i - p_j} - \sum_{j \in I_{k-1}} \frac{c_k c_{k-1}}{p_i - p_j}.$$

For this to make sense we need to assume that $p_j \neq p_i$ whenever $i \neq j$ and $|\ell_i - \ell_j| \leq 1$. We say the configuration is *nonsingular*. A more compact way to write F_i is to define, for any $i \in I$ and $1 \leq k \leq N$, the charge $Q_{i,k}$ by

$$Q_{i,k} = \begin{cases} -c_{\ell_i} & \text{if } \ k = \ell_i \\ c_{\ell_i} & \text{if } \ k = \ell_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then for $i \in I$,

$$F_i = \sum_{k=1}^{N} \sum_{j \neq i} \frac{Q_{i,k} Q_{j,k}}{p_i - p_j}.$$

One may think of p_i as a particle living in an N-sheeted plane and $Q_{i,k}$ as the charge of p_i in the k^{th} sheet. Then one may interpret $\overline{F}_i$ as an electrostatic force. The forces satisfy the following two basic equations

$$\sum_{i \in I} F_i = 0, \tag{1}$$

$$\sum_{i \in I} p_i F_i = \sum_{k=1}^{N} \sum_{i} \sum_{j < i} Q_{i,k} Q_{j,k}. \tag{2}$$

In terms of the neck sizes c_k, the second term in (2) is equal to

$$\mathcal{W} := \sum_{k=1}^{N-1} n_k (n_k - 1) c_k^2 - \sum_{k=1}^{N-2} n_k n_{k+1} c_k c_{k+1}. \tag{3}$$

Definition 1. We say the configuration is balanced if $\forall i \in I$, $F_i = 0$.

By (2), a necessary condition is that $\mathcal{W} = 0$. This gives one restriction on the neck sizes. The balancing condition is invariant by transformations of the form $p_i \mapsto ap_i + b$, where $a, b \in \mathbb{C}$, namely translation and complex scaling. Hence we may normalise two positions. The balancing condition is then a set of $n-2$ algebraic equations in $n-2$ variables.

Definition 2. We say the configuration is nondegenerate if the differential of $p = (p_i)_{i\in I} \mapsto F = (F_i)_{i\in I}$ has complex rank $n-2$.

This is the maximal rank it may have. Indeed, the invariance by translation and complex scaling gives that $(1, \dots, 1) \in \mathbb{C}^n$ and p are in its kernel. These vectors are independent unless $n = 1$, which we shall exclude.

1.4 Main results

In Section 3 we prove:

Theorem 1. *Consider a nonsingular, balanced, and nondegenerate configuration. Assume moreover that the differential of* $(c_1, \dots, c_{N-1}) \mapsto \mathcal{W}$ *has rank* 1*, i.e.,* $\partial\mathcal{W}/\partial c_k \neq 0$ *for at least one* k*. Then there exists a smooth family* $(M_t)_{0<t<\varepsilon}$ *of complete, unbranched minimal surfaces of finite total curvature, whose asymptotic behaviour when* $t \to 0$ *yields the given configuration: the necks at level* k *converge, after suitable translations, to catenoids of size* c_k*, and* M_t *scaled by* t *converges to an* N*-sheeted horizontal plane with* n *singular points at* p_i*,* $i \in I$*.* M_t *has genus* $n - N + 1$ *and* N *embedded ends whose logarithmic growths converge when* $t \to 0$ *to*

$$Q_k := \sum_{i\in I} Q_{i,k} = n_{k-1}c_{k-1} - n_k c_k, \qquad 1 \leq k \leq N.$$

Moreover, if $Q_1 < \cdots < Q_N$ *then* M_t *is embedded for* t *small enough.*

When this last condition is satisfied we say the configuration is *embedded*. See Section 3.9 for a more detailed geometric description of M_t. We will see in the proof of this theorem that the balancing condition is necessary for the existence of the family M_t, as well as the fact that necks at the same level have the same limit size (which is implicit in the definition of a configuration).

We will in fact prove the existence of a family of minimal surfaces depending on t and $N-2$ other parameters. These parameters are the logarithmic growths of the ends: if $\partial\mathcal{W}/\partial c_k \neq 0$, we may prescribe (locally) the logarithmic growths of all ends except the ends at levels k and $k+1$.

In Section 2 we discuss examples of configurations and prove:

Theorem 2. *There exists a configuration which is nonsingular, balanced, nondegenerate, embedded, and has no nontrivial symmetries.*

Combining these two theorems we obtain:

Corollary 1. *There exist minimal surfaces in $\mathbb{R}^3$ which are complete, properly embedded, have finite total curvature and no nontrivial symmetries.*

1.5 Example: The Costa Hoffman Meeks configuration

To recover the Costa Hoffman Meeks family, take $N=3$, $n_1=1$ and $n_2=m\geq 2$. Label the necks from 0 to m so that $\ell_0=1$ and $\ell_1=\cdots=\ell_m=2$. To find the neck sizes we use (3). We may normalise $c_2=1$. Then $\mathcal{W}=0$ gives $c_1=m-1$.

Take $p_0=0$ and $p_i=\omega^i$, $1\leq i\leq m$, where $\omega=\exp(2\pi\mathrm{i}/m)$. By symmetry we have $F_0=0$ and $p_iF_i=p_1F_1$, $1\leq i\leq m$. Hence (2) implies that $F_1=0$, so the configuration is balanced. We will see in Section 2.1 that it is nondegenerate.

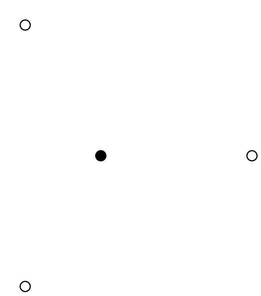

Figure 2: The Costa Hoffman Meeks configuration, $m=3$.

Note that Theorem 1 only recovers the existence of the Costa Hoffman Meeks family for large values of the parameter x.

The limit of the logarithmic growths of the ends are $Q_1=1-m$, $Q_2=-1$ and $Q_3=m$. So the configuration is embedded only if $m\geq 3$. In the case $m=2$, Theorem 1 does not guarantee that the surface is embedded. It is known to be embedded, but the proof is more difficult in this case (see [5], Remark 4.3).

1.6 Acknowledgements

As was already said, the method in this paper is inspired from the work of W. Meeks, J. Perez and A. Ros [8]. The idea to interpret algebraic equations as forces is reminiscent of the paper [1] by A. Douady and R. Douady. I would like to thank M. Wolf and S. Wolpert for suggesting the reference [7], and M. Weber for bringing [4] to my attention.

2. Examples

According to a theorem of R. Schoen, the only embedded minimal surface with finite total curvature and two ends is the catenoid. Therefore we should not expect any interesting configuration with $N = 2$. Indeed in this case,

$$\mathcal{W} = n_1(n_1 - 1)c_1^2 = 0$$

implies that $n_1 = 1$ so the configuration is trivial.

In Section 2.1, we classify all balanced configurations with $N = 3$ ends. We only find the Costa Hoffman Meeks configurations except in the genus 2 case where we also find a quite annoying degenerate configuration. Nevertheless, this gives some support to the conjecture that the only embedded minimal surfaces with 3 ends are the Costa Hoffman Meeks family (see [5], Section 5.2).

It is easy to compute balanced configuration with arbitrary number of ends and the same symmetries as the Costa Hoffman Meeks configurations. We will not discuss this here because embedded minimal surfaces with arbitrary number of ends have been constructed by several authors [6], [11]. It seems more interesting to investigate configurations with as little symmetries as possible.

In Section 2.3, we give numerical examples of embedded configurations with 4 ends, genus 8 and no nontrivial symmetries. Proving without any numerical computation that these configurations have no nontrivial symmetries seems hard.

In Section 2.4, we study in details a family of configurations with 4 ends and one nontrivial symmetry. As a corollary we obtain embedded minimal surfaces whose catenoidal ends have distinct axes. This answers a question raised in [5], Section 5.2.

In Section 2.5, we study configurations which are close to being singular: under suitable hypotheses, a singular configuration may be perturbed into a nonsingular configuration by perturbing the neck sizes.

In Section 2.6, we combine the results of Sections 2.4 and 2.5 to obtain configurations with no symmetries.

2.1 Classification of examples with 3 ends

Assume the number of ends is $N = 3$. Without loss of generality we may assume that $n_1 \leq n_2$ and $c_2 = 1$. Equation (3) gives

$$n_1(n_1 - 1)c_1^2 + n_2(n_2 - 1) - n_1 n_2 c_1 = 0,$$

$$\Delta = n_1 n_2(-3n_1 n_2 + 4n_1 + 4n_2 - 4).$$

If $n_1 \geq 2$ and $n_2 \geq 3$, then $\Delta < 0$ so the above equation has no real solution. Hence we are left with two cases:

- $n_1 = 1$, $n_2 \geq 2$ and $c_1 = (n_2 - 1)$,
- $n_1 = n_2 = 2$ and $c_1 = 1$.

We treat each case separately.

Proposition 1. *Assume that $n_1 = 1$, $n_2 = m \geq 2$, $c_1 = (m - 1)$ and $c_2 = 1$. Then the only balanced configuration is, up to translation, complex scaling and permutation of the points at each level, the Costa Hoffman Meeks configuration defined in the introduction. Moreover, the Costa Hoffman Meeks configuration is nondegenerate.*

Proof. Assume we have a balanced configuration $p_0, \ldots, p_m$. We may assume by translation that $p_0 = 0$. Then

$$F_i = \sum_{j \neq i, 0} \frac{2}{p_i - p_j} - \frac{(m-1)}{p_i - 0}, \quad 1 \leq i \leq m.$$

Let

$$P(z) = \prod_{i=1}^{m} (z - p_i).$$

A straightforward computation gives, since the zeros of P are simple,

$$\frac{P''(p_i)}{P'(p_i)} = \sum_{j \neq i} \frac{2}{p_i - p_j}.$$

Hence $F_i = 0$ is equivalent to

$$p_i P''(p_i) - (m-1)P'(p_i) = 0, \quad 1 \leq i \leq m.$$

Hence

$$zP''(z) - (m-1)P'(z) \equiv 0.$$

(Proof: this polynomial has m distinct zeros and has degree $\leq m-1$, so it is zero.) This gives by integration $P(z) = Az^m + B$ which proves that up to complex scaling, the configuration is the Costa Hoffman Meeks configuration.

This argument is inspired from Heine and Stieltjes who gave an electrostatic interpretation of the zeros of certain classical polynomials [4]. This was explained to me by Matthias Weber.

The Costa Hoffman Meeks configuration is nondegenerate. Our uniqueness proof can be adapted to prove non-degeneracy. This is simpler than a matrix computation.

Let $p_i(t)$ be a deformation of the Costa Hoffman Meeks configuration such that $\dot{F}_i(0) = 0$ (the dot means derivative with respect to t). We may normalise translation and scaling by $p_0(t) = 0$ and $p_1(t) \times \cdots \times p_m(t) = 1$. The goal is to prove that $\dot{p}_i(0) = 0$. Let

$$P_t(z) = \prod_{i=1}^{m} (z - p_i(t)).$$

Then $F_i = o(t)$ gives

$$p_i(t)P_t''(p_i(t)) - (m-1)P_t'(p_i(t)) = o(t).$$

Hence

$$zP_t''(z) - (m-1)P_t'(z) = o(t)$$

in the sense that the coefficients of this polynomial are $o(t)$. (Proof: the operator $L_t : \mathbb{C}_{m-1}[z] \to \mathbb{C}^m$ defined by

$$L_t(Q) = (Q(p_1(t)), \dots, Q(p_m(t)))$$

is an isomorphism and the norm of its inverse is bounded independently of t for t small enough. Hence $L_t(Q) = o(t)$ implies $Q = o(t)$.) Write

$$P_t(z) = \sum_{k=0}^{m} a_k(t) z^k.$$

We get

$$k(k-1)a_k(t) - (m-1)ka_k(t) = o(t).$$

Hence $a_k(t) = o(t)$ if $1 \leq k \leq m-1$, so $P_t(z) = P_0(z) + o(t)$ by the normalisation. Since the zeros are simple, they depend analytically on the coefficients, so $p_i(t) = p_i(0) + o(t)$. This proves that the Costa Hoffman Meeks configuration is nondegenerate. q.e.d.

Proposition 2. *Assume that $n_1 = n_2 = 2$ and $c_1 = c_2 = 1$. Label the necks from 1 to 4 so that $\ell_1 = \ell_2 = 1$ and $\ell_3 = \ell_4 = 2$. Then the only balanced configurations are, up to translation and complex scaling,*

$$p_1 = 1, \quad p_2 = -1, \quad p_3 = \alpha, \quad p_4 = 1/\alpha$$

where $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$ is a free parameter.

Proof. This is an elementary computation. We omit the details. q.e.d.

This configuration has a nontrivial free parameter α so it is degenerate and we cannot tell whether the corresponding family of minimal surfaces exists or not. In the most symmetrical case, it is proven not to exist in [10]. This non existing surface is usually called the Horgan surface. Actually the fact that the configuration is balanced explains why it is possible to make really good computer pictures of it. Observe that

$$\mathcal{W}(c_1, c_2) = 2(c_1 - c_2)^2 \Rightarrow d\mathcal{W}(1,1) = 0$$

so this configuration also fails to satisfy the last hypothesis of Theorem 1.

2.2 A differential equation equivalent to the balancing condition

Consider a configuration $\{p_i\}_{i \in I}$ with N ends. Let

$$P_k(z) = \prod_{i \in I_k} (z - p_i).$$

Following the proof of the uniqueness of the Costa Hoffman Meeks configurations, we want to write the balancing condition in function of the coefficients of the polynomials $P_1, \ldots, P_{N-1}$ instead of the points p_i. Write

$$P(z) = \prod_{k=1}^{N-1} P_k(z) = \prod_{i \in I} (z - p_i).$$

It will be convenient to write $P_0 = P_N = 1$. Observe that if $i \in I_k$,

$$\frac{P'_{k\pm 1}(p_i)}{P_{k\pm 1}(p_i)} = \sum_{j\in I_{k\pm 1}} \frac{1}{p_i - p_j}$$

$$\frac{P''_k(p_i)}{P'_k(p_i)} = \sum_{j\in I_k, j\neq i} \frac{2}{p_i - p_j}$$

$$F_i = c_k^2 \frac{P''_k}{P'_k} - c_k c_{k-1} \frac{P'_{k-1}}{P_{k-1}} - c_k c_{k+1} \frac{P'_{k+1}}{P_{k+1}} \quad \text{evaluated at } z = p_i.$$

Assume now that all points p_i are distinct, so $P'_k \times P/P_k$ is nonzero at p_i. Multiplying by $P'_k \times P/P_k$, $F_i = 0$ is equivalent to

$$c_k^2 P''_k \frac{P}{P_k} - c_k c_{k-1} P'_{k-1} P'_k \frac{P}{P_{k-1}P_k} - c_k c_{k+1} P'_k P'_{k+1} \frac{P}{P_k P_{k+1}} = 0 \quad \text{at } z = p_i.$$

So the configuration is balanced if and only if

$$\sum_{k=1}^{N-1} c_k^2 P''_k \frac{P}{P_k} - \sum_{k=1}^{N-2} c_k c_{k+1} P'_k P'_{k+1} \frac{P}{P_k P_{k+1}} \equiv 0. \tag{4}$$

(Proof: again this polynomial vanishes at the n points p_i, and has degree $\leq n-2$, so it is identically zero). Equation (4) is a system of algebraic equations with unknowns the coefficients of the polynomials $P_1, \ldots, P_{N-1}$. Observe that the coefficient of the highest order term in (4) is

$$\sum_{k=1}^{N-1} n_k(n_k - 1)c_k^2 - \sum_{k=1}^{N-2} n_k n_{k+1} c_k c_{k+1}$$

so we recover Equation (3).

Equation (4) does not have the geometrical flavour of the equation $F_i = 0$ but is much easier do deal with algebraically. The reason for this is clear: to one solution of (4) correspond $n_1! \ldots n_{N-1}!$ configurations by permutation of the points at each level.

One word of caution: (4) is equivalent to the balancing condition only if all the points p_i are distinct. For instance, (4) always has the trivial solution $P_k = z^{n_k}$ where all points p_i are equal to 0. Such unwanted solutions will be ruled out by suitable normalisation in the examples we will consider.

2.3 Numerical examples

Equation (4) is easy to solve numerically for small values of the numbers of ends and necks. Figure 3 shows a beautiful example in the case $N = 4$, $n_1 = 1$, $n_2 = 7$, $n_3 = 3$. The neck sizes are $c_1 = 27/7$, $c_2 = 1$, $c_3 = 5/2$. The logarithmic growths are $Q_1 = -27/7$, $Q_2 = -22/7$, $Q_3 = -1/2$, $Q_4 = 15/2$. The corresponding minimal surfaces have 4 ends, genus 8, are embedded and have no nontrivial symmetry.

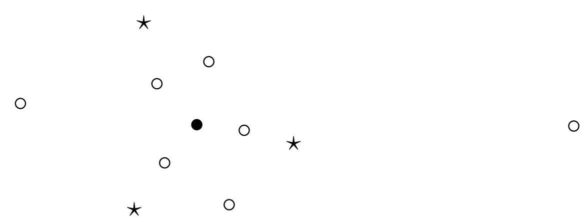

Figure 3: A numerical configuration of type $(1, 7, 3)$ with no symmetries. The dots, circles and stars represent the necks at level 1, 2 and 3 respectively.

Figure 4 shows another example in the case $N = 4$, $n_1 = 1$, $n_2 = 4$, $n_3 = 6$. The neck sizes are $c_1 = 7/3$, $c_2 = 1$, $c_3 = 2/3$. The logarithmic growths are $Q_1 = -7/3$, $Q_2 = -5/3$, $Q_3 = 0$, $Q_4 = 4$.

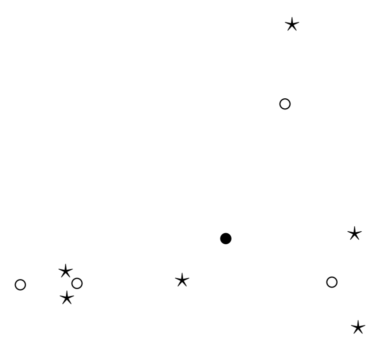

Figure 4: A numerical configuration of type $(1, 4, 6)$ with no symmetries.

As far as mathematical proof is concerned, these examples may be studied along the lines of the next section. One can prove that for generic values of the neck sizes, these configurations are nonsingular and nondegenerate. The hard point is to prove that they have no symmetries. This cannot be obtained by a genericity argument (because these configurations are symmetric for many values of c_3). To decide whether the configuration is symmetric or not, one needs to compute quite explicitly the coefficient of the polynomials involved, but this cannot reasonably be done by hand.

2.4 Configurations of type $(1, m, 2)$

Assume that $N = 4$, $n_1 = 1$, $n_2 = m \geq 2$ and $n_3 = 2$. We may assume by scaling that $c_2 = 1$. Equation (3) gives

$$c_1 = m - 1 - 2c_3 + \frac{2c_3^2}{m} \tag{5}$$

We study the configurations depending on the free parameter c_3 (in general, configurations with N ends depend on $N-3$ real parameters). Consider a configuration $p_0, \dots, p_{m+2}$ and assume that these points are distinct. We label the necks so that $\ell_0 = 1$, $\ell_1 = \cdots = \ell_m = 2$ and $\ell_{m+1} = \ell_{m+2} = 3$. We may assume by scaling and translation that $p_{m+1}p_{m+2} = 1$ and $p_0 = 0$. Let $X = p_{m+1} + p_{m+2}$. Write

$$P_2(z) = \prod_{i=1}^{m}(z - p_i) = \sum_{k=0}^{m} a_k z^k,$$

$$P_3(z) = (z - p_{m+1})(z - p_{m+2}) = z^2 - Xz + 1.$$

By (4), the configuration is balanced if and only if P_2 satisfies the differential equation

$$(z^3 - Xz^2 + z)P_2'' - \big(c_1(z^2 - Xz + 1) + c_3(2z^2 - Xz)\big)\, P_2' + 2c_3^2 z\, P_2 \equiv 0.$$

The coefficient of z^{m+1} is zero by (5). Looking at the coefficient of z^{k+1} for $0 \leq k \leq m-1$ gives

$$\begin{aligned}&\big(k(k-1) + 2c_3^2 - c_1 k - 2c_3 k\big)\, a_k \\ &= (k+1)(k - c_1 - c_3)X a_{k+1} - (k+2)(k+1-c_1)a_{k+2}.\end{aligned} \tag{6}$$

Provided the coefficients involved are nonzero (which is true for generic values of c_3), this determines $a_{m-1}, \dots, a_0$ by descending induction on k, starting with $a_m = 1$ and $a_{m+1} = 0$. This gives $a_k = A_k(X)$, $0 \leq k \leq$

$m-1$, where $A_k(X)$ is a polynomial of degree $m-k$ in the variable X with the same parity as $m-k$. The coefficients of $A_k(X)$ are rational functions of c_3.

The coefficient of z^0 in the differential equation is $-c_1 a_1$, so the configuration is balanced if and only if X satisfies $A_1(X)=0$.

Let us for example carry out the computation explicitly in the case $m=3$, $c_3=2$. Equation (5) gives $c_1=2/3$. Equation (6) gives

$$a_2=-3X, \qquad a_1=\frac{-6}{5}+3X^2, \qquad a_0=\frac{X}{4}+\frac{8}{3}Xa_1.$$

Solving $a_1=0$ gives $X=\pm\sqrt{10}/5$. With $X=\sqrt{10}/5$ we obtain

$$P_2=z^3-\frac{3\sqrt{10}}{5}z^2+\frac{\sqrt{10}}{20}$$
$$P_3=z^2-\frac{\sqrt{10}}{5}z+1.$$

Solving $P_2(z)=0$ and $P_3(z)=0$ gives, scaling all points by $\sqrt{10}$,

$$p_0=0, \quad p_1=1, \quad p_2=\frac{5+3\sqrt{5}}{2},$$
$$p_3=\frac{5-3\sqrt{5}}{2}, \quad p_4=1+3\mathrm{i}\,, \quad p_5=1-3\mathrm{i}\,.$$

This configuration has only one nontrivial symmetry. The logarithmic growths are $Q_1=-2/3$, $Q_2=-7/3$, $Q_3=-1$ and $Q_4=4$, so unfortunately it is not embedded. In fact we will see in Proposition 6 that a configuration of type $(1,m,2)$ cannot be embedded if $m\le 8$.

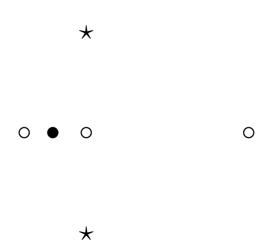

Figure 5: A configuration of type $(1,3,2)$ with $c_3=2$.

Returning to the general case, we need to prove that the configuration is nonsingular and nondegenerate.

Proposition 3. *For generic values of c_3, the following is true: for each X such that $A_1(X) = 0$, the points $p_0, \dots, p_{m+2}$ are distinct.*

Proof. Using the resultant, this statement may be written in the form $f(c_3) \neq 0$, f a rational function. Either $f \equiv 0$ or f has only a finite number of zeros. Hence it suffices to prove that the statement is true for at least one value of c_3. Unfortunately there seems to be no particular value of c_3 for which the coefficients of A_1 are easy to compute in function of m, except $c_3 = 0$ which is a singular value. So we prove the statement is true when $c_3 \to 0$, $c_3 \neq 0$. Let $x = c_3$. Using (6) with $k = m - 1$ and $k = m - 2$ gives

$$(m-1)a_{m-1} \simeq -mxX$$
$$2(m-2)a_{m-2} \simeq mx(-X^2+2).$$

For $1 \leq k \leq m-3$,

$$k(m-k)a_k \simeq (k+1)(m-k-1)Xa_{k+1} - (k+2)(m-k-2)a_{k+2}.$$

And for $k = 0$,

$$2x^2\, a_0 \simeq -X(m-1)a_1 + 2(m-2)a_2.$$

This implies that for $1 \leq j \leq m-1$, the limit

$$\alpha_j = \lim_{x \to 0} \frac{j(m-j)a_{m-j}}{mx}$$

exists and satisfies the induction formulae

$$\alpha_1 = -X, \quad \alpha_2 = -X^2 + 2, \quad \alpha_j = X\alpha_{j-1} - \alpha_{j-2}.$$

Hence $\alpha_j = -2T_j(X/2)$, where T_j is the j^{th} classical Chebyschev polynomial. Let X be a zero of A_1. For $x \to 0$, $X/2$ is close to a zero of T_{m-1}. By the above formula we have $A_k(X) = \mathcal{O}(x)$ if $1 \leq k \leq m-1$, and $A_0(X) \simeq \lambda/x$, with $\lambda = -mT_{m-2}(X/2)$. Since two consecutive Chebyschev polynomials have no common zeros, $\lambda \neq 0$. Hence the zeros of P_2 are equivalent when $x \to 0$ to the zeros of $z^m + \lambda x^{-1}$, so they are distinct and go to ∞ when $x \to 0$. Since ± 1 is not a zero of the Chebyschev polynomials, we have $X \neq \pm 2$ for x small enough, hence the zeros of P_3 are simple, nonzero, and bounded when $x \to 0$. Hence the points $p_0, \dots, p_{m+2}$ are distinct if x is small enough. q.e.d.

Proposition 4. *For generic values of c_3, the polynomial $A_1(X)$ has simple zeros.*

Proof. Again, it suffices to prove that this is true for at least one value of c_3. Since the Chebyschev polynomials have simple zeros, this is true for c_3 small enough. q.e.d.

Proposition 5. *For generic values of c_3, the configuration is nondegenerate.*

Proof. The proof is similar to the Costa Hoffman Meeks case. Consider a zero X_0 of A_1. By Proposition 4 we may assume that X_0 is simple. Let $p_0, \ldots, p_{m+2}$ be the corresponding configuration. Consider a deformation $p_0(t), \ldots, p_{m+2}(t)$ such that $\forall i$, $\dot{F}_i = 0$. We may normalise by $p_0(t) = 0$ and $p_{m+1}(t)p_{m+2}(t) = 1$. Let $X_t = p_{m+1}(t) + p_{m+2}(t)$. Define the polynomials $P_{2,t}$ and $P_{3,t}$ in the obvious way. Then $F_i = o(t)$ gives $A_1(X_t) = o(t)$. Observe that c_3 is fixed here so the polynomial A_1 does not depend on t. Since X_0 is simple, this gives $X_t = X_0 + o(t)$. Hence $P_{2,t} = P_2 + o(t)$ and $P_{3,t} = P_3 + o(t)$. This gives $\forall i$, $p_i(t) = p_i + o(t)$. q.e.d.

Proposition 6. *The configuration is embedded if and only if*

$$\frac{m}{4} < c_3 < \frac{m - \sqrt{2m}}{2} \quad \text{or} \quad \frac{m + \sqrt{2m}}{2} < c_3 < \sqrt{\frac{m(m+1)}{2}}.$$

When $m \leq 8$, the above intervals are empty so the configuration is never embedded.

Proof. This is a straightforward computation. We omit the details. The values $m/4$, $(m \pm \sqrt{2m})/2$ and $\sqrt{m(m+1)/2}$ correspond respectively to the equality cases $Q_3 = Q_4$, $Q_1 = Q_2$ and $Q_2 = Q_3$. q.e.d.

Define $\mu_1, \ldots, \mu_4$ by

$$\mu_k = \left(\sum_{i=0}^{m+2} Q_{i,k} p_i \right) \bigg/ \left(\sum_{i=0}^{m+2} Q_{i,k} \right)$$

where the charges $Q_{i,k}$ are defined in the introduction.

Proposition 7. *For generic values of c_3 the following is true. For each zero of $A_1(X)$ if m is odd and of $A_1(X)/X$ if m is even, the numbers $\mu_1, \ldots, \mu_4$ are distinct.*

Corollary 2. *There exists embedded minimal surfaces whose four catenoidal ends have distinct axes.*

Proof. The numbers $\mu_1, \ldots, \mu_4$ determine the axes of the ends. See Proposition 15. q.e.d.

Proof of the Proposition. It suffices to prove that this is true for at least one value of c_3, and we prove this is true for $c_3 = x \to 0$. Using the computations of Proposition 3, we have

$$\sum_{i=1}^{m} p_i = -a_{m-1} \simeq \frac{mxX}{m-1}.$$

This gives

$$\mu_1 = 0, \qquad \mu_2 \simeq \frac{mxX}{m-1}, \qquad \mu_3 \simeq \frac{xX}{m(m-1)}, \qquad \mu_4 = \frac{X}{2}.$$

q.e.d.

Proposition 8. *Let X_1, X_2 be two zeros of $A_1(X)$. If $X_1 = \pm X_2$ (resp. $X_1 = \pm\overline{X_2}$), then the corresponding configurations are congruent by $z \mapsto \pm z$ (resp. $z \mapsto \pm\overline{z}$). Conversely if the two configurations are congruent by a transformation $z \mapsto az + b$ (resp. $z \mapsto a\overline{z} + b$), then $X_1 = \pm X_2$ (resp. $X_1 = \pm\overline{X_2}$).*

Proof. The proof is straightforward. We omit the details. (It is understood that a congruence preserves the level of the necks.) q.e.d.

Take $X_1 = X_2 = X$ in this proposition. Let Γ be the symmetry group of the configuration.

- If $X = 0$, then $\Gamma = \{z \mapsto \pm z,\ z \mapsto \pm\overline{z}\}$,
- if $X \in \mathbb{R}^*$, then $\Gamma = \{\mathrm{id},\ z \mapsto \overline{z}\}$,
- if $X \in \mathrm{i}\,\mathbb{R}^*$ then $\Gamma = \{\mathrm{id},\ z \mapsto -\overline{z}\}$,
- if $X \notin \mathbb{R}$ and $X \notin \mathrm{i}\,\mathbb{R}$ then $\Gamma = \{\mathrm{id}\}$.

Numerically, it seems that the zeroes of $A_1(X)$ are always either real or imaginary, but this is hard to prove. Indeed, one cannot use genericity arguments as above, so one has to compute explicitly the coefficients of $A_1(X)$.

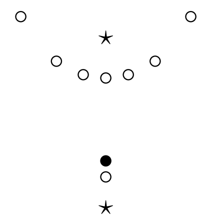

Figure 6: Configuration of type $(1, 9, 2)$ with $c_3 = 20/3$, $X \simeq 1.02223115\,\mathrm{i}$.

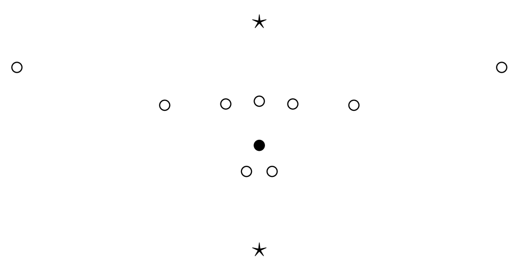

Figure 7: Configuration of type $(1, 9, 2)$ with $c_3 = 20/3$, $X \simeq .17786964\,\mathrm{i}$.

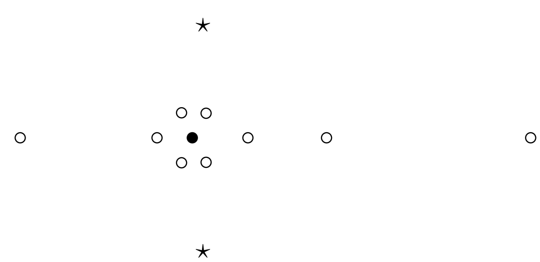

Figure 8: Configuration of type $(1, 9, 2)$ with $c_3 = 20/3$, $X \simeq .18731052$.

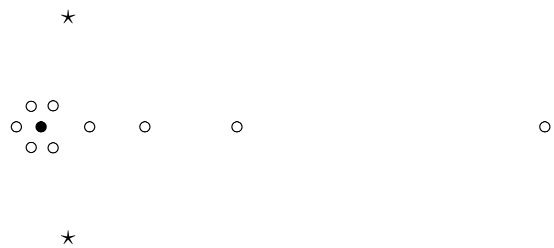

Figure 9: Configuration of type $(1, 9, 2)$ with $c_3 = 20/3$, $X \simeq .47637921$.

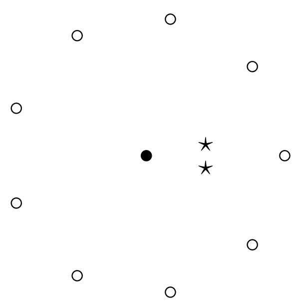

Figure 10: Configuration of type $(1, 9, 2)$ with $c_3 = 10^{-3}$, $X \simeq 1.961678$.

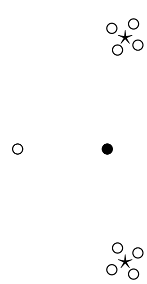

Figure 11: Configuration of type $(1, 9, 2)$ with $c_3 = 3.001$, $X \simeq 0.315153$.

If all the zeroes of $A_1(X)$ are simple and either real or imaginary, the number of non-congruent configurations that we obtain is $[m/2]$.

Pictures. Figures 6 to 9 show the four configurations we obtain in the case $m=9$, $c_3=20/3$. This is in the range of Proposition 6 so these configurations yield embedded minimal surfaces. We will use these configurations in Section 2.6 to construct a configuration with no nontrivial symmetries.

Figure 10 shows a configuration with $m=9$ and c_3 close to 0. This illustrates the computation in the proof of Proposition 3.

Figure 11 shows a configuration with $m=9$ and c_3 close to 3. Observe that the two groups of five points look like small copies of the genus 3 Costa Hoffman Meeks configuration. When $c_3\to 3$, the configuration converges to a singular configuration with four points, two of which have multiplicity 5. This suggests a method to construct balanced configurations: start with a singular configuration, and perturb the neck sizes to obtain a nonsingular configuration. This is the subject of the next section.

2.5 Perturbation of a singular configuration

In this section we consider configurations $p_{i,\mu}$ of the form

$$p_{i,\mu}=\widehat{p}_i+\lambda_i\widetilde{p}_{i,\mu},\qquad 1\leq i\leq n,\ \ 1\leq\mu\leq m_i$$

where λ_i are complex numbers close to zero, so the configuration is close to be singular. $I=\{(i,\mu)\mid 1\leq i\leq n,\ 1\leq\mu\leq m_i\}$ is a set of multi-indices used to label the necks.

Assume we have a family of such configurations, depending on the neck sizes $c=(c_1,\dots,c_{N-1})$ in a neighborhood of $c^0=(c_1^0,\dots,c_{N-1}^0)$. Assume that $\lambda_i=0$ when $c=c^0$, so the configuration $p_{i,\mu}^0$ is singular. Define charges $Q_{i,\mu,k}$ in function of c as in the introduction. Then

$$\begin{aligned}F_{i,\mu}&=\sum_k\sum_{(j,\nu)\neq(i,\mu)}\frac{Q_{i,\mu,k}Q_{j,\nu,k}}{p_{i,\mu}-p_{j,\nu}}\\&=\frac{1}{\lambda_i}\widetilde{F}_{i,\mu}+\sum_k\sum_{j\neq i}\sum_\nu\frac{Q_{i,\mu,k}Q_{j,\nu,k}}{\widehat{p}_i-\widehat{p}_j}+\mathcal{O}(\lambda)\end{aligned}$$

where

$$\widetilde{F}_{i,\mu}=\sum_k\sum_{\nu\neq\mu}\frac{Q_{i,\mu,k}Q_{i,\nu,k}}{\widetilde{p}_{i,\mu}-\widetilde{p}_{i,\nu}}.$$

Assume that the configuration $(p_{i,\mu})_{(i,\mu)\in I}$ is balanced for all c. In the limit $c \to c^0$, we obtain $\widetilde{F}_{i,\mu} = 0$, so each sub-configuration $(\widetilde{p}^{\,0}_{i,\mu})_{1\le\mu\le m_i}$ is balanced.

Remark 1. Together with our classification of examples with three ends, this explains why the Costa Hoffman Meeks configuration shows up in Figure 11. Quite interesting, the Horgan configuration (Proposition 2) often appears as a sub-configuration.

From (1) and the above equation we obtain

$$\sum_\mu \widetilde{F}_{i,\mu} = 0 \Rightarrow \sum_\mu F_{i,\mu} = \widehat{F}_i + \mathcal{O}(\lambda)$$

where

$$\widehat{F}_i = \sum_k \sum_{j\neq i} \frac{Q_{i,k} Q_{j,k}}{\widehat{p}_i - \widehat{p}_j}, \qquad Q_{i,k} = \sum_\mu Q_{i,\mu,k}.$$

In the limit $c \to c^0$ we obtain $\widehat{F}_i = 0$ so the configuration $(\widehat{p}^{\,0}_i)_{1\le i\le n}$ is balanced. This is very similar to the dipole computation of classical electrostatics. We can also compute the equivalent of λ_i when $c \to c^0$ using

$$\sum_\mu \widetilde{p}_{i,\mu} F_{i,\mu} = \frac{1}{\lambda_i}\widetilde{\mathcal{W}}_i + \Lambda_i + \mathcal{O}(\lambda)$$

where, using (2)

$$\widetilde{\mathcal{W}}_i = \sum_\mu \widetilde{p}_{i,\mu}\widetilde{F}_{i,\mu} = \sum_k \sum_\mu \sum_{\nu<\mu} Q_{i,\mu,k} Q_{i,\nu,k}$$

$$\Lambda_i = \sum_k \left(\sum_\mu Q_{i,\mu,k}\widetilde{p}_{i,\mu}\right)\left(\sum_{j\neq i}\frac{Q_{j,k}}{\widehat{p}_i - \widehat{p}_j}\right).$$

This gives $\lambda_i(c) \simeq -\widetilde{\mathcal{W}}_i(c)/\Lambda_i$, provided $\Lambda_i \neq 0$.

Our objective in this section is to go backwards. Assume we are given neck sizes $(c^0_1, \dots, c^0_{N-1})$, n sub-configurations $\widetilde{p}^{\,0}_{i,\mu}$ and a configuration $\widehat{p}^{\,0}_i$. We want to recover a family of balanced configurations $p_{i,\mu}$ depending on the parameter $c = (c_1, \dots, c_{N-1})$. We assume that $\widehat{p}^{\,0}_1, \dots, \widehat{p}^{\,0}_n$ are distinct, and $\widetilde{p}^{\,0}_{i,1}, \dots, \widetilde{p}^{\,0}_{i,m_i}$ are distinct for all i. Some of the sub-configurations may have only one point, in which case we say they are trivial. Let n' be the number of nontrivial configurations. We assume that the sub-configurations $\widetilde{p}^{\,0}_{i,\mu}$ are nontrivial ($m_i \ge 2$) if

$1 \leq i \leq n'$ and trivial ($m_i = 1$) if $n'+1 \leq i \leq n$. We use vector notations

$$p = (p_{i,\mu})_{(i,\mu)\in I}, \qquad \widehat{p} = (\widehat{p}_1, \dots, \widehat{p}_n), \qquad \lambda = (\lambda_1, \dots, \lambda_{n'})$$
$$\widetilde{p}_i = (\widetilde{p}_{i,3}, \dots, \widetilde{p}_{i,m_i}), \qquad \widetilde{p} = (\widetilde{p}_1, \dots, \widetilde{p}_{n'}).$$

The variables in this construction are $(c, \widetilde{p}, \widehat{p}, \lambda)$. We fix the value of the remaining parameters as follows: for nontrivial sub-configurations we take $\widetilde{p}_{i,1} = \widetilde{p}^0_{i,1}$ and $\widetilde{p}_{i,2} = \widetilde{p}^0_{i,2}$, for trivial sub-configurations we take $\widetilde{p}_{i,1} = 0$ and $\lambda_i = 0$. Then by a straightforward application of the Inverse Function Theorem, $(\widetilde{p}, \widehat{p}, \lambda) \mapsto p$ is a diffeomorphism in a neighborhood of any point such that $\lambda_i \neq 0$ for all $i \leq n'$. Let

$$\mathcal{W} = \sum_{i,\mu} p_{i,\mu} F_{i,\mu} = \sum_k \sum_{i,\mu} \sum_{(j,\nu)<(i,\mu)} Q_{i,\mu,k} Q_{j,\nu,k}.$$

Theorem 3. *Assume that all given configurations are balanced, namely*

$$\forall i \leq n' \quad \forall \mu \quad \widetilde{F}_{i,\mu}(c^0, \widetilde{p}^0_i) = 0 \quad \textit{and} \quad \forall i \leq n \quad \widehat{F}_i(c^0, \widehat{p}^0) = 0$$

and nondegenerate. Assume that

$$\forall i \leq n' \quad \Lambda_i(c^0, \widetilde{p}^0_i, \widehat{p}^0) \neq 0.$$

Then there exists analytic maps $\widetilde{p}(c)$, $\widehat{p}(c)$ and $\lambda(c)$, defined in a neighborhood of c^0, such that $\widetilde{p}(c^0) = \widetilde{p}^0$, $\widehat{p}(c^0) = \widehat{p}^0$, $\lambda(c^0) = 0$, and for any c in a neighborhood of c^0 the following is true: if $\mathcal{W}(c) = 0$ and $\lambda_i(c) \neq 0$ for all $i \leq n'$, the configuration $p_{i,\mu}(c) = \widehat{p}_i(c) + \lambda_i(c)\widetilde{p}_{i,\mu}(c)$ is nonsingular, balanced (namely $F_{i,\mu}(c, p(c)) = 0$), and nondegenerate. Moreover, we have

$$\lambda_i(c) = \frac{\widetilde{\mathcal{W}}_i(c)}{\Lambda_i(c^0, \widetilde{p}^0_i, \widehat{p}^0)} + o(c - c^0)$$

which may be used to guarantee that $\lambda_i \neq 0$.

Remark 2. If a sub-configuration is invariant by a nontrivial rotation, we may assume that the rotation fixes the origin, then

$$\forall k \quad \sum_\mu Q_{i,\mu,k} \widetilde{p}^0_{i,\mu} = 0$$

which implies that $\Lambda_i = 0$. So we cannot use this theorem in the case of sub-configuration with rotational symmetry, such as the Costa Hoffman Meeks configurations. A result is possible in this case but one has to look at higher order terms and this is quite technical.

Proof. Define functions of the variables $(c, \widetilde{p}, \widehat{p}, \lambda)$, $\lambda_i \neq 0$, by

$$\mathcal{F}_{i,\mu} = \lambda_i F_{i,\mu}, \quad \mathcal{G}_i = \sum_{\mu} F_{i,\mu}, \quad \mathcal{H}_i = \sum_{\mu} \lambda_i \widetilde{p}_{i,\mu} F_{i,\mu}$$

By the above computation, these functions extend analytically to $\lambda_i = 0$, with

$$\lambda_i = 0 \quad \Rightarrow \quad \mathcal{F}_{i,\mu} = \widetilde{F}_{i,\mu}, \quad \mathcal{G}_i = \widehat{F}_i, \quad \mathcal{H}_i = \widetilde{\mathcal{W}}_i.$$

Let

$$\mathcal{F} = (\mathcal{F}_{i,\mu})_{1 \leq i \leq n',\, 3 \leq \mu \leq m_i} \quad \mathcal{G} = (\mathcal{G}_i)_{3 \leq i \leq n} \quad \mathcal{H} = (\mathcal{H}_i)_{1 \leq i \leq n'}.$$

The value of the map $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ at $(c^0, \widetilde{p}^0, \widehat{p}^0, 0)$ is zero. Its partial differential with respect to $(\widetilde{p}, \widehat{p}, \lambda)$ at this point has the form

$$\begin{pmatrix} \mathrm{Diag}(\partial \widetilde{F}_i / \partial \widetilde{p}_i) & 0 & \cdot \\ 0 & \partial \widehat{F} / \partial \widehat{p} & \cdot \\ 0 & 0 & \mathrm{Diag}(\Lambda_i) \end{pmatrix}.$$

By non-degeneracy, $\partial \widetilde{F}_i / \partial \widetilde{p}_i$ is an isomorphism (thanks to the fact that we took $\mu \geq 3$ in the definition of $\mathcal{F}_{i,\mu}$ and $\widetilde{p}_i$) and $\partial \widehat{F} / \partial \widehat{p}$ is surjective with a 2-dimensional kernel (see Remark 6). By the implicit function theorem, there exists analytic maps $\widetilde{p}(c)$, $\widehat{p}(c)$, $\lambda(c)$, such that $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is zero at $(c, \widetilde{p}(c), \widehat{p}(c), \lambda(c))$.

We now prove that if $\mathcal{W}(c) = 0$, and $\lambda_i(c) \neq 0$, the configuration $p_{i,\mu}(c)$ is balanced. From (1) and (2) we have

$$0 = \sum_{i,\mu} F_{i,\mu} = \sum_i \mathcal{G}_i$$

$$0 = \mathcal{W}(c) = \sum_{i,\mu} p_{i,\mu} F_{i,\mu} = \sum_i \widehat{p}_i \mathcal{G}_i + \sum_i \mathcal{H}_i.$$

Hence

$$\mathcal{G}_1 + \mathcal{G}_2 = 0,\ \widehat{p}_1 \mathcal{G}_1 + \widehat{p}_2 \mathcal{G}_2 = 0 \ \Rightarrow\ \forall i,\ \mathcal{G}_i = 0.$$

If $i \leq n'$, we have

$$F_{i,1} + F_{i,2} = \mathcal{G}_i = 0,\ \lambda_i \widetilde{p}_{i,1} F_{i,1} + \lambda_i \widetilde{p}_{i,2} F_{i,2} = \mathcal{H}_i = 0 \ \Rightarrow\ \forall \mu,\ F_{i,\mu} = 0.$$

If $n' + 1 \leq i \leq n$, we have $F_{i,1} = \mathcal{G}_i = 0$. So the configuration is balanced.

It remains to prove it is nondegenerate. By continuity, the partial differential of $(\mathcal{F},\mathcal{G},\mathcal{H})$ remains surjective when c is in a neighborhood of c^0. Since $(\widetilde{p}\,,\widehat{p}\,,\lambda)\mapsto p$ is a local diffeomorphism, the partial differential of $(\mathcal{F},\mathcal{G},\mathcal{H})$ with respect to p at $p(c)$ is surjective, so it has a two dimensional kernel. Write D for the partial differential with respect to p at $p(c)$. Let X be a vector in the kernel of DF. Then $D\mathcal{F}_{i,\mu}(X)=0$, $D\mathcal{G}_i(X)=0$ and $D\mathcal{H}_i(X)=0$. Hence DF has a kernel of dimension at most 2, so the configuration is nondegenerate. q.e.d.

2.6 An embedded non-symmetric configuration

It is now rather clear that configurations with no symmetries should exist: consider the Costa Hoffman Meeks configuration and replace the necks at level 2 by small copies of the configurations of Section 2.4. Theorem 3 guarantees that this may be perturbed into a balanced configuration. If the sub-configurations are non-congruent the resulting configuration will have no symmetries. In this way, non-symmetry is a consequence of non-uniqueness.

Proposition 9. *There exists a configuration with $N=5$, $n_1=4$, $n_2=36$, $n_3=8$, $n_4=1$, which is nonsingular, balanced, nondegenerate, embedded, and has no nontrivial symmetries.*

Proof. We use Theorem 3 using four sub-configurations of type $(1,9,2)$ and one sub-configuration consisting of one single point at level 4. We choose the sub-configurations $\widetilde{p}^{\,0}_{i,\mu}$, $i\leq 4$, as follows. As in Section 2.4 we take $c_2^0=1$, c_3^0 is a free parameter and c_1^0 is determined by solving $\widetilde{\mathcal{W}}_i=0$, so c_1^0 is given in function of c_3^0 by (5). For each $i\leq 4$ we choose a nonzero number X_i such that $A_1(X_i)=0$ and let $\widetilde{p}^{\,0}_{i,\mu}$ be the corresponding configuration.

The configuration $\widehat{p}^{\,0}_1,\dots,\widehat{p}^{\,0}_5$ is determined as follows. A necessary condition for $\widehat{F}_i=0$ is

$$\widehat{\mathcal{W}}=\sum_k\sum_{i<j}Q_{i,k}Q_{j,k}=0$$

which determines c_4^0 in function of the other neck sizes. Then since all sub-configurations have the same number of necks and neck sizes, we have $Q_{1,k}=Q_{2,k}=Q_{3,k}=Q_{4,k}$. Hence $\widehat{p}^{\,0}_i$ must be the Costa Hoffman Meeks configuration:

$$\widehat{p}^{\,0}_1=1,\quad \widehat{p}^{\,0}_2=\mathrm{i}\,,\quad \widehat{p}^{\,0}_3=-1,\quad \widehat{p}^{\,0}_4=-\mathrm{i}\,,\quad \widehat{p}^{\,0}_5=0,$$

and it is non degenerate.

For generic values of c_3^0, the following is true: if $X_i \neq 0$ then

$$\Lambda_i(c^0, \widetilde{p}_i^{\,0}, \widehat{p}^{\,0}) \neq 0$$

Indeed, as in the Proposition 7, it suffices to prove that this is true for one value of c_3^0. We find, after some computations similar to those of Proposition 7, when $c_3^0 \to 0$,

$$\Lambda_i \simeq -\frac{3(m^2-m+1)\,X_i}{2\widehat{p}_i^{\,0}} \neq 0 \qquad \text{(where } m=9\text{)}.$$

Consider a value of c_3^0 such that all generic statements we have seen are true. Theorem 3 gives the existence of a family of (possibly singular) configurations $p_{i,\mu}(c)$ for c in a neighborhood of c^0. Take $c_2 = c_2^0 = 1$, $c_3 = c_3^0$, $c_1 = c_1^0 + x$ where x is a free parameter. We find c_4 in function of x by solving $\mathcal{W} = 0$. Then $\widetilde{\mathcal{W}}_i = -9x$ so $\lambda_i = 9x/\Lambda_i + o(x) \neq 0$ when $x \neq 0$ is small enough. By Theorem 3, the configuration $p_{i,\mu}$ is nonsingular, balanced, and nondegenerate when x is small enough.

Embeddedness. If we take $c_3^0 = 20/3$ and $x = 0$ we find

$$c_1 = \frac{368}{81}, \qquad c_4 = \frac{777673}{29160}$$
$$Q_1 = \frac{-1472}{81}, \quad Q_2 = \frac{-1444}{81}, \quad Q_3 = \frac{-1404}{81},$$
$$Q_4 = \frac{777527}{29160}, \quad Q_5 = \frac{777673}{29160}$$

so the configuration is embedded. This remains true if c_3^0 is a generic value close enough to $20/3$ and x is small enough.

Symmetries. One of the following alternatives holds:

- All zeros of $A_1(X)$ are real or pure imaginary, in which case we may choose $X_1, \ldots, X_4$ so that the corresponding sub-configurations are non-congruent (see the discussion after Proposition 8) hence the configuration $p_{i,\mu}$ has no nontrivial symmetries.
- $A_1(X)$ has a zero, say X_1, which is neither real nor pure imaginary, in which case we can take $X_2 = X_3 = X_4$ different from $\pm X_1$, $\pm\overline{X_1}$, and the configuration again has no symmetries (numerically, this case does not seem to happen).

q.e.d.

Remark 3. In the same way we can use n' configurations of type $(1, m, 2)$ with $n' \geq 2$ and $m \geq 3$. We obtain non-symmetric configurations provided $n' \geq 3$ and $m \geq 7$, and embedded configurations provided $n' \geq 4$ and $m \geq 9$.

3. Proof of Theorem 1

We use the Weierstrass Representation of minimal surfaces, which may be written

$$X_1(z) + \mathrm{i}\, X_2(z) = \frac{1}{2}\left(\overline{\int_{z_0}^{z} g^{-1}dh} - \int_{z_0}^{z} g dh\right), \qquad X_3(z) = \mathrm{Re}\int_{z_0}^{z} dh$$

where $z \in \Sigma$, Σ is a Riemann Surface, g is a meromorphic function (the Gauss map) and dh is a holomorphic 1-form on Σ (usually called the height differential, but dh is not exact). $X = (X_1, X_2, X_3) : \Sigma \to \mathbb{R}^3$ is well defined provided

$$\overline{\int_{\gamma} g^{-1}h} - \int_{\gamma} g dh = 0 \quad \text{and} \quad \mathrm{Re}\int_{\gamma} dh = 0$$

for any cycle γ on Σ: this is the Period Problem. X is regular (i.e., an immersion) with embedded ends provided the divisors of g and dh satisfy some well known conditions: we call them the zero/pole equations. A good reference on Weierstrass Representation is [5].

We define (Σ, g, dh) depending on the parameter $t > 0$ (which is the same as t in the statement of Theorem 1) and some other parameters. Σ and g are defined by explicit formulae. dh is defined in a more abstract way by prescribing its residues and periods on the cycles γ_i of a canonical homology basis γ_i, Γ_i. The key point is that when $t \to 0$, we can compute explicitly the limit of dh. Roughly speaking, when $t \to 0$, the Riemann surface Σ degenerates into a Riemann surface with nodes, whose parts have genus zero. Using Algebraic Geometry results, we prove that dh converges on each part to a meromorphic differential with simple poles which we can compute explicitly (because the genus is zero).

Using our explicit formulae for the Weierstrass data when $t \to 0$, we compute the limit of the zero/pole equation ands the periods of the Weierstrass data. After suitable re-normalisation, we prove that each equation extends smoothly to $t = 0$. We solve the equations when $t = 0$.

We solve the equations for t in a neighbourhood of 0 using the Implicit Function Theorem at $t = 0$.

Once this is done, we have a well defined minimal immersion $X : \Sigma \to \mathbb{R}^3$, depending on the remaining parameter t. Using our asymptotic formulae for the Weierstrass data when $t \to 0$, we prove that this minimal surface satisfies all geometric conclusions of Theorem 1, in particular we prove embeddedness.

3.1 The Gauss map

It will be convenient to assume that the set used to label the necks is $I = \{1, \ldots, n\}$. The parameters needed to define the Gauss map are $t > 0$ and $4n$ complex numbers $a_i, b_i, \alpha_i, \beta_i$, $1 \leq i \leq n$. Consider N copies of the complex plane, labelled $\mathbb{C}_1, \ldots, \mathbb{C}_N$. Define a meromorphic function g on the disjoint union $\mathbb{C}_1 \cup \ldots \cup \mathbb{C}_N$ by

$$g(z) = \begin{cases} t g_k(z) & \text{if } z \in \mathbb{C}_k,\ k \text{ odd} \\ (t g_k(z))^{-1} & \text{if } z \in \mathbb{C}_k,\ k \text{ even} \end{cases}$$

$$g_k(z) = \sum_{i \in I_k} \frac{\alpha_i}{z - a_i} + \sum_{i \in I_{k-1}} \frac{\beta_i}{z - b_i}.$$

Here we see a_i, $i \in I_k$ and b_i, $i \in I_{k-1}$ as points in $\mathbb{C}_k$, and we assume these points are distinct. We also assume α_i and β_i are nonzero.

To define Σ, we identify pairs of points z, z' in $\mathbb{C}_1 \cup \ldots \cup \mathbb{C}_N$ to create necks. The points we identify should satisfy $g(z) = g(z')$ so that g is well defined in Σ.

Consider some $i \in I_k$. Use $v_i = 1/g_k$ and $w_i = 1/g_{k+1}$ as local complex coordinates in a neighborhood of a_i and b_i respectively. If $z \in \mathbb{C}_k$ and $z' \in \mathbb{C}_{k+1}$ are respectively in a neighborhood of a_i and b_i, and k is, say, odd,

$$g(z) = g(z') \iff \frac{t}{v_i(z)} = \frac{w_i(z')}{t} \iff v_i(z) w_i(z') = t^2.$$

So the definition of Σ is as follows. Take $\mathbb{C}_1 \cup \ldots \cup \mathbb{C}_N$. Consider a fixed small enough $\varepsilon > 0$. For each $i \in I$, remove the disks $|v_i| \leq t^2/\varepsilon$ and $|w_i| \leq t^2/\varepsilon$. Identify the points z and z' such that

$$\frac{t^2}{\varepsilon} < |v_i(z)| < \varepsilon, \qquad \frac{t^2}{\varepsilon} < |w_i(z')| < \varepsilon, \qquad v_i(z) w_i(z') = t^2.$$

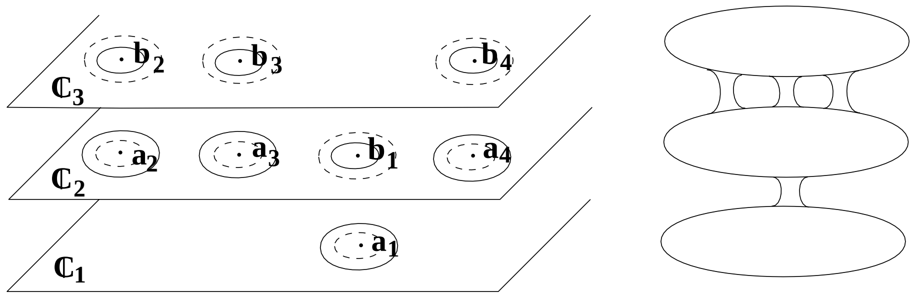

Figure 12: Left: definition of Σ. Each circle is identified with the circle above it (plain with plain, dots with dots). Right: topological picture of $\widehat{\Sigma}$.

This defines a Riemann Surface Σ (see Figure 12). This is the conformal model for the minimal surface we want to construct. By construction g is a well defined meromorphic function on Σ. Let $\widehat{\Sigma} = \Sigma \cup \{\infty_1, \ldots, \infty_N\}$ be the compactification of Σ, where ∞_k is the point at infinity in $\mathbb{C}_k$. We have $g(\infty_k) = 0$ if k is odd and $g(\infty_k) = \infty$ if k is even.

3.2 The height differential

We need to define a meromorphic 1-form dh with (at most) simple poles at $\infty_1, \ldots \infty_N$. Let me recall some standard complex analysis. Let $\widehat{\Sigma}$ be a compact Riemann Surface of genus G.

- A canonical homology basis of $\widehat{\Sigma}$ is a set of $2G$ closed curves γ_i, Γ_i, $1 \leq i \leq G$ such that γ_i intersects Γ_i with intersection number 1 and all other intersection numbers are zero.

- The space of holomorphic 1-forms on $\widehat{\Sigma}$ has complex dimension G. An isomorphism with $\mathbb{C}^G$ is given by integration along the curves $\gamma_1, \ldots, \gamma_G$.

- The space of meromorphic 1-forms with simple poles at given points $q_1, \ldots, q_m$ has complex dimension $G + m - 1$. One may prescribe the integrals along the curves $\gamma_1, \ldots \gamma_G$ and the residues at the poles, with the only condition that the sum of the residues be zero.

In our case, it follows from the topological picture that $\widehat{\Sigma}$ has genus $n - N + 1$. We define a canonical basis as on Figure 13. γ_i is a small circle around a_i with the negative orientation. It is homologous in Σ to a small circle around b_i with the positive orientation. We think of γ_i as a curve around the i^{th} neck. A formal definition of Γ_i will be given in Section 3.6 when we compute the periods along Γ_i of the Weierstrass data. Let $i_0(k) = \min I_k$ and $J = \{i \in I \mid i > i_0(\ell_i)\}$. Then $\{\gamma_i, \Gamma_i\}_{i \in J}$ is a canonical homology basis of $\widehat{\Sigma}$.

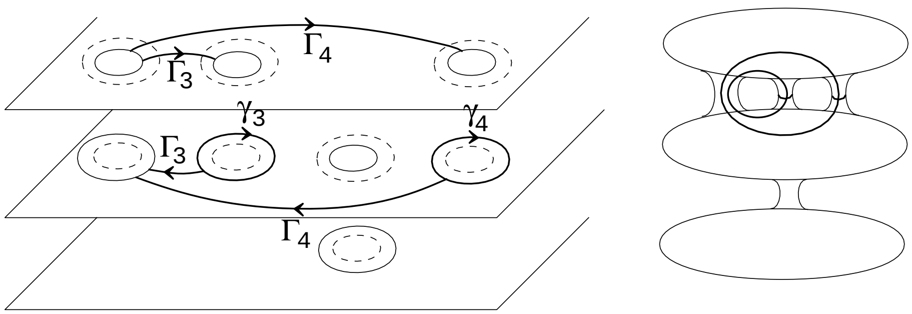

Figure 13: canonical homology basis (genus 2).

We define the height differential dh as the unique meromorphic 1-form on Σ with simple poles at $\infty_1, \dots, \infty_N$ such that

$$\forall i \in J, \quad \int_{\gamma_i} dh = 2\pi \mathrm{i}\, \mathrm{r}_i$$

$$\forall k, 1 \le k \le N, \quad \mathrm{Res}_{\infty_k} dh = -\mathrm{R}_k$$

where r_i, $i \in J$ are positive real numbers and $\mathrm{R}_1, \dots, \mathrm{R}_N$ are real numbers whose sum is zero. Geometrically, r_i is the size of the i^{th} neck and R_k is the logarithmic growth of the catenoidal end ∞_k. If $i \notin J$, we *define* r_i by $\int_{\gamma_i} dh = 2\pi \mathrm{i}\, \mathrm{r}_i$. Then

$$\forall k, \quad \sum_{i \in I_k} \mathrm{r}_i - \sum_{j \in I_{k-1}} \mathrm{r}_j = -\mathrm{R}_k. \tag{7}$$

Proof. Use the Residue Theorem in the domain $\Omega_k \subset \mathbb{C}_k$ defined in Section 3.3, and observe that homologically speaking,

$$\partial \Omega_k = \sum_{i \in I_k} \gamma_i - \sum_{i \in I_{k-1}} \gamma_i$$

3.3 Parameters

We use vector notations for the parameters: $a = (a_1, \dots, a_n)$, $\mathrm{r} = (\mathrm{r}_i)_{i \in J}$ and so on. What we have done so far is define triples (Σ, g, dh) depending on $t > 0$ and the parameters $\alpha, \beta, a, b, \mathrm{r}, \mathrm{R}$. When needed we write $\mathbf{X}$ for the collection of all parameters. Let p_i^0, c_k^0 and Q_k^0 be the neck positions, sizes and logarithmic growths of the given configuration. We will solve the period problem using the Implicit Function Theorem at the point $\mathbf{X}^0$ defined by

$$t^0 = 0, \qquad \mathrm{R}_k^0 = Q_k^0$$
$$\forall i \in I_k, \quad -\alpha_i^0 = \beta_i^0 = \mathrm{r}_i^0 = c_k^0$$
$$\forall i \in I_k, \quad a_i^0 = -\overline{b_i^0} = \begin{cases} \overline{p_i^0} & \text{if } k \text{ odd} \\ -p_i^0 & \text{if } k \text{ even.} \end{cases}$$

Let Ω_k be the domain in $\mathbb{C}_k$ defined by $\forall i \in I_k$, $|z - a_i^0| > \epsilon$ and $\forall i \in I_{k-1}$, $|z - b_i^0| > \epsilon$, where ϵ is a fixed small number. If $\mathbf{X}$ is close enough to $\mathbf{X}^0$, the disks that were removed when defining Σ are outside Ω_k, so we may see Ω_k as a domain in Σ. Note that Ω_k does not depend on any parameter, and all surfaces Σ have the domain Ω_k in common.

The restriction of dh to Ω_k depends analytically on all parameters, in the sense that if $z \in \Omega_k$, the function $(z, \mathbf{X}) \mapsto dh(z)/dz$ is analytic. Analytic dependence of Abelian differentials on moduli is a classical problem. This may be proven using, for instance, Theta functions. From a modern Algebraic Geometry point of view this is a consequence of coherent sheaf results (Grauert's Semi-continuity Theorem). In the next section we will see that the map $(z, \mathbf{X}) \mapsto dh$ extends analytically to $t = 0$.

Remark 4. The parameters (t, α, β, a, b) are not moduli for the couples (Σ, g) because different values of the parameters may give isomorphic (Σ, g). To see this, fix some k and let

$$a_i' = A a_i + B, \quad \alpha_i' = A\alpha_i, \quad i \in I_k$$
$$b_i' = A b_i + B, \quad \beta_i' = A\beta_i, \quad i \in I_{k-1}$$

where A, B are complex numbers, $A \neq 0$. Let g_k', g' and Σ' be the corresponding objects. Then $g_k'(Az+B) = g_k(z)$ so the map $\varphi : \Sigma \to \Sigma'$ defined by $z \mapsto Az+B$ in $\mathbb{C}_k$ is a well defined isomorphism and $g' \circ \varphi = g$. As a consequence, we may normalise complex scaling and translation in $\mathbb{C}_k$ by fixing the value of certain parameters. We will do this when needed.

3.4 The height differential extends analytically to $t = 0$

In this section we write $\Sigma = \Sigma_t$ and $dh = dh_t$ to emphasise the dependence of the Riemann Surface and the height differential on t. All other parameters have fixed value.

Proposition 10. *When $t \to 0$, dh_t converges uniformly on compacts subsets of Ω_k to*

$$\sum_{i \in I_k} \frac{-\mathrm{r}_i\, dz}{z - a_i} + \sum_{i \in I_{k-1}} \frac{\mathrm{r}_i\, dz}{z - b_i}.$$

Proof. To prove the proposition we follow Fay [2] and Masur [7]. The situation is as follows. Consider N Riemann spheres $S_1, \dots S_N$ and $2n$ distinct points $a_1, \dots, a_n, b_1, \dots, b_n$ in the disjoint union $S_1 \cup \dots \cup S_N$. Consider some fixed local complex coordinates v_i and w_i in a neighborhood of a_i and b_i respectively, $v_i(a_i) = w_i(b_i) = 0$. Given some small complex number $s \neq 0$ we define Σ_s by removing the disks $|v_i| \leq |s|$ and $|w_i| \leq |s|$ and identifying points z and z' such that for some i,

$$|s| < |v_i(z)| < 1, \quad |s| < |w_i(z')| < 1, \quad v_i(z) w_i(z') = s.$$

We assume that the coordinates v_i and w_i are chosen so that the above annular regions are disjoint. This defines a compact Riemann Surface Σ_s. (Compare with Section 3.2: t was real and the identification rule was $v_i w_i = t^2$.)

The first step is to see each Σ_s as the level set of a holomorphic function $f : \mathcal{S} \to \mathbb{C}$, i.e., $\Sigma_s = f^{-1}(s)$, $\mathcal{S}$ a 2-dimensional complex manifold. From an Algebraic Geometry point of view this is precisely the definition of a holomorphic family of complex curves. This is the meaning of the sentence: Σ_s depends holomorphically on s.

Let Ω be the domain $|v_i| \geq \frac{1}{2}$, $|w_i| \geq \frac{1}{2}$ of $S_1 \cup \dots \cup S_N$. Let D be the unit disk in $\mathbb{C}$. Let $U_1, \dots, U_n$ be n copies of $\{(v, w, s) \in D^3 \mid vw = s\}$. (Note that this is a regular surface in $\mathbb{C}^3$, which is the reason why we write $v_i w_i = s$ instead of $v_i w_i = t^2$.)

Let $\mathcal{S} = (\Omega \times D) \cup (U_1 \cup \dots \cup U_n)$ where we identify a point $(z, s) \in \Omega \times D$ such that $\frac{1}{2} < |v_i(z)| < 1$ with the point $(v_i(z), s/w_i(z), s)$ of U_i, and in the same way we identify a point $(z, s) \in \Omega \times D$ such that $\frac{1}{2} < |w_i(z)| < 1$ with the point $(s/w_i(z), w_i(z), s)$ of U_i. This defines a complex analytic 2-manifold. We define a holomorphic function $f : \mathcal{S} \to D$ by $f(z, s) = s$ on $\Omega \times D$ and $f(v, w, s) = s = vw$ on U_i.

Σ_s is clearly isomorphic to $f^{-1}(s)$ if $s \neq 0$, so we may identify $\Sigma_s = f^{-1}(s)$. The level set $f^{-1}(0)$ is a singular complex curve. It has an ordinary double point at the point $(0,0,0)$ of each U_i. From the Riemann Surface point of view, we may see $\Sigma_0 = f^{-1}(0)$ has $S_1 \cup \ldots \cup S_N$ with a_i identified with b_i, $1 \leq i \leq n$. The point $a_i = b_i$ is called a node. Σ_0 is a Riemann Surface with nodes.

Let G be the genus of Σ_s, $s \neq 0$. Consider a canonical homology basis $\gamma_1, \ldots, \gamma_G, \Gamma_1, \ldots, \Gamma_G$ as in Section 3.2, namely, such that all cycles γ_i may be represented by fixed (i.e., independent of s) circles in Ω. Fay proves the following ([2], Proposition 3.7)

For each i, $1 \leq i \leq G$, there exists a unique holomorphic 2-form ω_i on $\mathcal{S}$ whose residue along each Σ_s, $s \neq 0$, is the unique holomorphic 1-form $\omega_{i,s}$, such that

$$\int_{\gamma_j} \omega_{i,s} = \delta_{i,j}.$$

In other words, $\omega_{i,s}$ is a normalised Abelian differential of the first kind on Σ_s. Masur proves the following ([7], Proposition 4.2):

Given two fixed points p, q in Ω, there exists a unique meromorphic 2-form $\omega_{p,q}$ on $\mathcal{S}$, which has a simple pole along $\{p\} \times D$ and $\{q\} \times D$, and whose residue along each Σ_s, $s \neq 0$, is the unique meromorphic 1-form $\omega_{p,q,s}$ which has simple poles at p and q with respective residues 1 and -1, and such that

$$\int_{\gamma_i} \omega_{p,q,s} = 0.$$

In other words, $\omega_{p,q,s}$ is a normalised Abelian differential of the third kind. Since any meromorphic 1-form with simple poles may be written as a linear combination of normalised Abelian differentials of the first and third kind, we obtain:

Given some fixed points $q_1, \ldots, q_m$ in Ω, complex numbers $\mathrm{R}_1, \ldots, \mathrm{R}_m$ such that $\mathrm{R}_1 + \cdots + \mathrm{R}_m = 0$, and complex numbers $\mathrm{r}_1, \ldots, \mathrm{r}_G$, there exists a unique meromorphic 2-form on $\mathcal{S}$ whose residue along each Σ_s, $s \neq 0$, is the unique meromorphic 1-form ω_s which has simple poles at each q_i, with residue R_i, and such that

$$\int_{\gamma_i} \omega_s = \mathrm{r}_i.$$

In term of local complex coordinates z_1, z_2 on $\mathcal{S}$, if we write $\omega = h(z_1, z_2)dz_1 \wedge dz_2$, the residue of ω along $\Sigma_{s_0} = f^{-1}(s_0)$ is the 1-form

$$\omega_{s_0} = \left.\frac{h(z_1, z_2)dz_1}{\partial f/\partial z_2}\right|_{\Sigma_{s_0}} = -\left.\frac{h(z_1, z_2)dz_2}{\partial f/\partial z_1}\right|_{\Sigma_{s_0}}.$$

In other words ω_{s_0} is the Poincaré residue of $\omega/(f - s_0)$ (see [3] page 147). This is easily seen to be independent of the chosen coordinates z_1, z_2.

If $z_0 \in \Omega$, we may use (z, s) as local coordinates on $\mathcal{S}$ in a neighborhood of (z_0, s_0). Since $f(z, s) = s$, this gives

$$\omega_{s_0} = h(z, s_0)dz.$$

From this formula we see that ω_s depends holomorphically on s. In a neighborhood of the point $(0, 0, 0) \in U_i$ we may use v, w as local coordinates. Since $f(v, w) = vw$ we obtain

$$\omega_0 = h(v, 0)\frac{dv}{v} = -h(0, w)\frac{dw}{w}.$$

Hence ω_0 has a simple pole at $v = 0$ in the component $w = 0$ of the set $vw = 0$, and a simple pole at $w = 0$ in the component $v = 0$, with opposite residues. So ω_0 has two simple poles at a_i and b_i.

Returning to the situation at hand, we see that dh_t depends holomorphically on $s = t^2$, hence on t, t in a neighborhood of 0. Restricting t to real numbers, we obtain that dh_t depends analytically on t. dh_0 is a meromorphic 1-form on $\mathbb{C}_1 \cup \ldots \cup \mathbb{C}_N$ with simple poles at $a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\infty_1, \ldots \infty_N$. From the definition of dh_t and by continuity we obtain

$$\mathrm{Res}_{a_i} dh_0 = -\mathrm{r}_i, \quad \mathrm{Res}_{b_i} dh_0 = \mathrm{r}_i, \quad \mathrm{Res}_{\infty_k} dh_0 = \mathrm{R}_k.$$

This determines dh_0 and completes the proof of Proposition 10. q.e.d.

3.5 The zero/pole equation

We recall the conditions that the zeros and poles of g and dh must satisfy.

1. The zeros of dh in Σ must be precisely the zeros and poles of g, with the same multiplicity.

2. At each end ∞_k, if g has a simple zero or pole, dh needs a simple pole, while if g has a zero or pole of multiplicity $m \geq 2$, then dh needs a zero of multiplicity $m-2$. This insures that the end is embedded (asymptotic to a half catenoid in the first case, and a plane in the other one).

Since dz has a double pole at ∞, these conditions are equivalent to: the zeros of dh/dz are precisely the zeros and poles of g (the zeros of dh/dz makes sense, although dz is not globally defined in Σ).

Proposition 11. *For $(t, a, b, \mathrm{r}, \mathrm{R})$ in a neighborhood of $(0, a^0, b^0, \mathrm{r}^0, \mathrm{R}^0)$, there exists $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, depending analytically on $(t, a, b, \mathrm{r}, \mathrm{R})$, such that the corresponding Weierstrass data satisfies Conditions 1 and 2 above. Moreover, when $t = 0$, we have $\alpha_i = -\mathrm{r}_i$ and $\beta_i = \mathrm{r}_i$.*

Proof. In this section we write $dh = dh_{\mathbf{X}}$, where $\mathbf{X} = (t, \alpha, \beta, a, b, \mathrm{r}, \mathrm{R})$ is the set of all parameters. The first step is to write Conditions 1 and 2 in the form $\mathcal{F}(\mathbf{X}) = 0$, $\mathcal{F}$ an analytic map in a neighborhood of $\mathbf{X}^0$.

The number of zeros of dh minus the number of poles, counting multiplicities, is equal to $2G-2$, G the genus of Σ. Hence the number of zeros of dh/dz is $2G-2+2N = 2(n-N+1)-2+2N = 2n$. Since the degree of g is $2n$, it suffices to prove that at each zero of dh/dz, g has a zero or pole, with greater or equal multiplicity (the multiplicities will then be equal and g will have no further zeros and poles by the above counting argument).

We first prove the proposition in the case where all zeros of $dh_{\mathbf{X}^0}/dz$, including ∞_k, are simple. Then for $\mathbf{X}$ close to $\mathbf{X}^0$, the zeros of $dh_{\mathbf{X}}/dz$ will be simple, and depend analytically on $\mathbf{X}$ (this is a consequence of the Weierstrass Preparation Theorem, see below). By Proposition 10, $dh_{\mathbf{X}^0}$ has $n_{k-1}+n_k-1$ finite zeros in $\mathbb{C}_k$. By taking ε small enough we may assume that all these zeros are in Ω_k. For $\mathbf{X}$ close to $\mathbf{X}^0$, $dh_{\mathbf{X}}$ will have $n_{k-1}+n_k-1$ zeros in Ω_k. Let $\zeta_{k,i}$ be these zeros, $1 \leq i \leq n_{k-1}+n_k-1$. Let

$$\mathcal{F}_{k,i}(\mathbf{X}) = g_k(\zeta_{k,i}), \qquad \mathcal{F}(\mathbf{X}) = (\mathcal{F}_{k,i}) \in \mathbb{C}^{2n-N}.$$

Conditions 1 and 2 are equivalent to $\mathcal{F}(\mathbf{X}) = 0$. $\mathcal{F}$ is an analytical map. From our choice of $\alpha_i^0 = -r_i^0$, $\beta_i^0 = r_i^0$, we have $dh = g_k dz$ when $\mathbf{X} = \mathbf{X}^0$. Hence $\mathcal{F}(\mathbf{X}^0) = 0$.

We normalise the parameters α, β as follows. For each $k \leq N-1$, choose an integer $i \in I_k$ and fix $\alpha_i = -r_i$. For $k = N$, choose an integer

$i \in I_{N-1}$ and fix $\beta_i = r_i$. This normalises scaling in $\mathbb{C}_k$ (see Remark 4).

Let L be the partial differential of $\mathcal{F}$ with respect to the remaining $2n - N$ variables α, β at $\mathbf{X}^0$. I claim that $L : \mathbb{C}^{2n-N} \to \mathbb{C}^{2n-N}$ is an isomorphism. To see this, assume that $(\alpha, \beta) \in \mathrm{Ker} L$. Note that when $t = 0$, dh does not depend on (α, β), so $\mathcal{F}$ is linear in the variables (α, β). Hence $L(\alpha, \beta) = 0$ means that for each k, the functions

$$\sum_{i \in I_k} \frac{-\mathrm{r}_i^0}{z - a_i^0} + \sum_{i \in I_{k-1}} \frac{\mathrm{r}_i^0}{z - b_i^0}$$

and

$$\sum_{i \in I_k} \frac{\alpha_i}{z - a_i^0} + \sum_{i \in I_{k-1}} \frac{\beta_i}{z - b_i^0}$$

have the same zeros. Since they have the same poles, they are proportional, so that

$$\alpha_i = -\lambda_k \mathrm{r}_i^0,\ i \in I_k, \quad \beta_j = \lambda_k \mathrm{r}_i^0,\ i \in I_{k-1}.$$

By normalisation, $\lambda_k = 0$. Hence L is an isomorphism.

By the Implicit Function Theorem, there exists a unique analytical map $(t, a, b, \mathrm{r}, \mathrm{R}) \mapsto (\alpha, \beta)$ defined in a neighborhood of $(0, a^0, b^0, \mathrm{r}^0, \mathrm{R}^0)$ such that $\mathcal{F}(t, \alpha, \beta, a, b, \mathrm{r}, \mathrm{R}) = 0$. The last statement of the proposition is a consequence of uniqueness and normalisation. This proves the proposition in the case where all zeros are simple.

In the case where $dh_{\mathbf{X}^0}$ has a multiple zero at some point $\zeta \in \mathbb{C}_k$, with multiplicity m, we modify the definition of $\mathcal{F}$ as follows. By the Weierstrass Preparation Theorem ([3] page 8), we may write, for z in a neighborhood of ζ and $\mathbf{X}$ in a neighborhood of $\mathbf{X}^0$,

$$dh_X = f(z, \mathbf{X}) P_{\mathbf{X}}(z) dz$$

where f does not vanish and $P_{\mathbf{X}}(z)$ is a Weierstrass Polynomial, namely a unitary z-polynomial of degree m whose coefficients are analytic functions of $\mathbf{X}$. From this we see that for $\mathbf{X}$ close to $\mathbf{X}^0$, dh has m zeros close to ζ, counting multiplicity. (In case $m = 1$, we also see that the simple zeros of dh depend analytically on $\mathbf{X}$, as claimed above). Let

$$Q_{\mathbf{X}}(z) = g_k(z) \prod_{i \in I_k} (z - a_i) \prod_{i \in I_{k-1}} (z - b_i).$$

This is a z-polynomial whose zeros are the zeros of g_k. Let $\mathcal{F}_\zeta(\mathbf{X}) \in \mathbb{C}_{m-1}[z] \simeq \mathbb{C}^m$ be the remainder of $Q_{\mathbf{X}}/P_{\mathbf{X}}$. Since $P_{\mathbf{X}}$ is unitary, $\mathcal{F}_\zeta$

is a polynomial function of the coefficients of $P_{\mathbf{X}}$ and $Q_{\mathbf{X}}$, hence an analytical function of $\mathbf{X}$.

If $dh_{\mathbf{X}^0}/dz$ has a zero of multiplicity $m \geq 2$ at ∞_k (which means that dh has no pole at ∞_k, so the residue $\mathrm{R}_k = 0$), we do the same thing using $w = 1/z$ as a local coordinate in a neighborhood of ∞_k. Note that whatever the value of the parameters, both dh/dz and g_k have a zero at ∞_k. By the Weierstrass Preparation Theorem, we may write, in a neighborhood of $w = 0$ and $\mathbf{X} = \mathbf{X}^0$,

$$\begin{aligned} \frac{dh_{\mathbf{X}}}{dz} &= f(w, \mathbf{X})\ w\ P_{\mathbf{X}}(w) \\ g_k &= \widetilde{f}(w, \mathbf{X})\ w\ Q_{\mathbf{X}}(w). \end{aligned}$$

Let $\mathcal{F}_{\infty_k}(\mathbf{X}) \in \mathbb{C}_{m-2}[z] \simeq \mathbb{C}^{m-1}$ be the remainder of $Q_{\mathbf{X}}/P_{\mathbf{X}}$. Let $\mathcal{F}(\mathbf{X}) \in \mathbb{C}^{2n-N}$ be the collection of all the maps defined above. The proof of Proposition 11 is exactly the same in this case. q.e.d.

3.6 Γ-periods of the height differential

We now start to solve the period problem.

Proposition 12. *Assume that α, β are given by Proposition 11. For (t, a, b, R) in a neighborhood of $(0, a^0, b^0, \mathrm{R}^0)$, there exists a unique $\mathrm{r} = (\mathrm{r}_i)_{i \in J}$ such that the corresponding Weierstrass data satisfies*

$$\forall i \in J, \quad \mathrm{Re} \int_{\Gamma_i} dh = 0.$$

Moreover, when $t = 0$, we have $\mathrm{r}_i = c_{\ell_i}$ where $c_1, \ldots, c_{N-1}$ are defined in function of $\mathrm{R}_1, \ldots, \mathrm{R}_N$ by $c_0 = 0$ and the induction formula

$$c_{k-1} n_{k-1} - c_k n_k = \mathrm{R}_k, \quad 1 \leq k \leq N-1.$$

The proof of this proposition is similar to the proof of Proposition 11. We first write the period condition in the form $\mathcal{F}(\mathbf{X}) = 0$ and then we use the Implicit Function Theorem at $\mathbf{X}^0$ (note that the $\mathcal{F}$ in this section has nothing to do with $\mathcal{F}$ in the previous section). To define $\mathcal{F}$ we need:

Lemma 1. *Consider some k, $1 \leq k \leq N-1$. Let $i_0 = i_0(k) = \min I_k$. Given $i \in I_k$ such that $i > i_0$, we have*

$$\int_{\Gamma_i} dh = (\mathrm{r}_i - \mathrm{r}_{i_0}) \log t^2 \ + \text{analytic}$$

where analytic *means a (bounded) analytic function of $\mathbf{X}$ in a neighborhood of $\mathbf{X}^0$.*

Proof. We first make more precise the definition of Γ_i. We define Γ_i as the composition of the following four paths:

1. A path from $v_i = \varepsilon/2$ to $v_{i_0} = \varepsilon/2$, contained in Ω_k, depending continuously on $\mathbf{X}$.
2. The path from $v_{i_0} = \varepsilon/2$ to $w_{i_0} = \varepsilon/2$ parametrised by
$$v_{i_0} = (1-s)\frac{\varepsilon}{2} + s\frac{2t^2}{\varepsilon}, \quad s \in [0,1].$$
3. A path from $w_{i_0} = \varepsilon/2$ to $w_i = \varepsilon/2$, contained in Ω_{k+1}, depending continuously on $\mathbf{X}$.
4. The path from $w_i = \varepsilon/2$ to $v_i = \varepsilon/2$ parametrised by
$$v_i = s\frac{\varepsilon}{2} + (1-s)\frac{2t^2}{\varepsilon}, \quad s \in [0,1].$$

Remark 5. Note that the second path stays in the annular region $t^2/\varepsilon < |v_{i_0}| < \varepsilon$ because t is real, so this defines a path on Σ (which goes through the neck). If t were, say, pure imaginary, this path would hit $v_{i_0} = 0$, which is not a point on Σ. In fact, when t is complex there is no way to define Γ_i as a continuous function of t, for if t^2 makes one turn around 0, Γ_i increases by γ_i. This multi-valuation is clear on the formula for $\int_{\Gamma_i} dh$ because of the $\log t^2$ term.

The integral of dh on the first and third paths is an analytic function of $\mathbf{X}$ because these paths are contained in Ω where dh is an analytic function of $(z, \mathbf{X})$. To estimate the integral of dh on the fourth path we write the Laurent series of dh in the annular region $t^2/\varepsilon < |v_i| < \varepsilon$
$$dh = \sum_{n\in\mathbb{Z}} a_n v_i^n dv$$
where a_n depends on all parameters (including t) and is given by
$$a_n = \frac{1}{2\pi\mathrm{i}}\int_{|v_i|=\varepsilon} \frac{dh}{v_i^{n+1}} = \frac{1}{2\pi\mathrm{i}}\int_{|v_i|=t^2/\varepsilon} \frac{dh}{v_i^{n+1}}.$$
From the first equality we see that a_n is an analytic function of $\mathbf{X}$. Since the circles $|v_i| = \varepsilon$ and $|w_i| = \varepsilon$ are included in Ω, we have
$$\int_{|v_i|=\varepsilon} |dh| \leq 2\pi C$$
$$\int_{|v_i|=t^2/\varepsilon} |dh| = \int_{|w_i|=\varepsilon} |dh| \leq 2\pi C$$

for some constant C. This gives the estimates

$$|a_n| \leq \frac{C}{\varepsilon^{n+1}}, \qquad |a_n| \leq C\left(\frac{\varepsilon}{t^2}\right)^{n+1}. \tag{8}$$

The first one is useful if $n \geq 0$, the second one if $n \leq -2$. Now we compute, if $t \neq 0$,

$$\begin{aligned}\int_{w_i=\varepsilon/2}^{v_i=\varepsilon/2} dh &= \int_{v=2t^2/\varepsilon}^{v=\varepsilon/2} \sum_{n\in\mathbb{Z}} a_n v^n dv \\ &= a_{-1}\log\frac{\varepsilon^2}{4t^2} + \sum_{n\neq -1} \frac{a_n}{n+1}\left((\varepsilon/2)^{n+1} - (2t^2/\varepsilon)^{n+1}\right).\end{aligned}$$

Using the above estimates we get, provided $|t| < \varepsilon/2$, that

$$\sum_{n\neq -1} \left|a_n(\varepsilon/2)^{n+1}\right| \leq \sum_{n<-1} C\left(\frac{\varepsilon^2}{2t^2}\right)^{n+1} + \sum_{n>-1} C\,(1/2)^{n+1} \leq C'$$

$$\sum_{n\neq -1} \left|a_n(2t^2/\varepsilon)\right|^{n+1} \leq \sum_{n<-1} C\,2^{n+1} + \sum_{n>-1} C(2t^2/\varepsilon^2)^{n+1} \leq C'$$

for some constant C'. Hence the second term in the integral of dh extends analytically to $t = 0$. (Proof: think of t as a complex number, then this is a well defined bounded holomorphic function if $t \neq 0$, so it extends holomorphically to $t = 0$ by the Riemann Extension Theorem in several complex variables [3], page 9). This gives

$$\int_{w_i=\varepsilon/2}^{v_i=\varepsilon/2} dh = -\mathrm{r}_i \log\frac{\varepsilon^2}{4t^2} + \text{analytic}.$$

The integral on the second path is evaluated in the same way. q.e.d.

Proof of Proposition 12. Assume that α and β are given by Proposition 11. Let

$$\mathcal{F}_i(t, a, b, \mathrm{r}, \mathrm{R}) = \frac{1}{\log t}\mathrm{Re}\int_{\Gamma_i} dh.$$

The problem is that $\mathcal{F}_i$ is not differentiable with respect to t at $t = 0$. We solve this problem by writing $t = \exp(-1/\tau^2)$ where τ is a real number in a neighborhood of 0. From now on our parameter is τ instead of t. Note that t is a smooth function of τ, with $t = 0$ when $\tau = 0$. Then

$$\mathcal{F}_i(\tau, a, b, \mathrm{r}, \mathrm{R}) = 2(\mathrm{r}_i - \mathrm{r}_{i_0}) - \tau^2 \times \text{analytic}(t, a, b, \mathrm{r}, \mathrm{R})$$

is a smooth function.

When $\tau = 0$ we have $\mathcal{F}_i = 2(\mathrm{r}_i - \mathrm{r}_{i_0})$. Hence $\mathcal{F}_i(X^0) = 0$. The partial differential of $(\mathcal{F}_i)_{i\in J}$ with respect to $(\mathrm{r}_i)_{i\in J}$ at $\mathbf{X}^0$ is an isomorphism from $\mathbb{R}^{n-N+1}$ to $\mathbb{R}^{n-N+1}$. By the Implicit Function Theorem, there exists a (unique) smooth map $(\tau, a, b, \mathrm{R}) \mapsto \mathrm{r} = (\mathrm{r}_i)_{i\in J}$ such that $\mathcal{F}(\tau, a, b, \mathrm{r}, \mathrm{R}) = 0$.

When $\tau = 0$, we have by uniqueness $\mathrm{r}_i = \mathrm{r}_{i_0}$, so all necks at the same level have the same size. Let c_k be the size of the necks at level k. By (7) we have

$$-n_k c_k + n_{k-1} c_{k-1} = \mathrm{R}_k.$$

This proves the last statement of Proposition 12. q.e.d.

3.7 Horizontal Γ-periods

We continue with the period problem. We define the horizontal period along a cycle c by

$$P(c) = \left(\overline{\int_c g^{-1} dh} - \int_c g dh\right).$$

Proposition 13. *Assume that α and β are given by Proposition 11 and r is given by Proposition 12. For (τ, a, R) in a neighborhood of $(0, a^0, \mathrm{R}^0)$, there exists $b = (b_1, \dots, b_n)$, depending smoothly on (τ, a, R), such that the corresponding Weierstrass data satisfies*

$$\forall i \in J, \quad P(\Gamma_i) = 0.$$

Moreover, when $\tau = 0$, we have $b_i = -\overline{a_i}$.

We need:

Lemma 2. *Assume that α and β are given by Proposition 11. Consider some k, $1 \le k \le N-1$. Let $i_0 = \min I_k$. Consider some $i \in I_k$ such that $i > i_0$. Then if k is odd (resp. even), $P(\Gamma_i)$ (resp. $-\overline{P(\Gamma_i)}$) is equal to*

$$t^{-1}(b_{i_0} + \overline{a_{i_0}} - b_i - \overline{a_i}) + \text{analytic} + t\log t \times \text{analytic}.$$

Proof. We see Γ_i as the composition of a path from $v_i = t$ to $v_{i_0} = t$, contained in Ω_k, and a path from $w_{i_0} = t$ to $w_i = t$, contained in Ω_{k+1}. (Think of the point $v_i = w_i = t$ as the middle of the neck.) First a fast computation. By Proposition 11, we have, when $t=0$, $dh = g_k dz$ in $\mathbb{C}_k$. If k is odd, we have, in $\mathbb{C}_k$

$$\overline{g^{-1}dh} - g\,dh = t^{-1}\overline{g_k^{-1}dh} - t\,g_k dh \overset{t\to 0}{\simeq} t^{-1}\overline{dz}. \tag{9}$$

If k even, we have in $\mathbb{C}_k$

$$\overline{g^{-1}dh} - g\,dh = t\,\overline{g_k dh} - t^{-1}g_k^{-1}dh \overset{t\to 0}{\simeq} -t^{-1}dz. \tag{10}$$

Hence if k is odd,

$$P(\Gamma_i) \simeq \int_{v_i=t}^{v_{i_0}=t} t^{-1}\overline{dz} + \int_{w_{i_0}=t}^{w_i=t} -t^{-1}dz \simeq t^{-1}(\overline{a_{i_0}} - \overline{a_i} - b_i + b_{i_0}).$$

If k is even,

$$P(\Gamma_i) \simeq \int_{v_i=t}^{v_{i_0}=t} -t^{-1}dz + \int_{w_{i_0}=t}^{w_i=t} t^{-1}\overline{dz} \simeq t^{-1}(-a_{i_0} + a_i + \overline{b_i} - \overline{b_{i_0}}).$$

These computations are of course not correct because (9) and (10) only hold on Ω_k, and not up to the middle of the necks. To correct these computations we use a Laurent series expansion of dh as in Section 3.6. Assume for example that k is odd and consider the integral from $v_i = t$ to $v_i = \varepsilon/2$.

$$\overline{\int_{v_i=t}^{v_i=\varepsilon/2} g^{-1}dh} - \int_{v_i=t}^{v_i=\varepsilon/2} g\,dh = \frac{1}{t}\left(\overline{\int_{v_i=t}^{v_i=\varepsilon/2} g_k^{-1}dh} - t^2\int_{v_i=t}^{v_i=\varepsilon/2} g_k dh\right).$$

With the notations of Section 3.6, we have in the domain $|t| < |v_i| < \varepsilon/2$,

$$g_k^{-1}dh = \sum_{n\in\mathbb{Z}} a_n v_i^{n+1}dv$$

$$t^2 g_k dh = \sum_{n\in\mathbb{Z}} t^2 a_n v_i^{n-1}dv.$$

Using the estimates (8), we have, if $0 < |t| < |v_i| < \varepsilon/2$

$$\sum_{n\in\mathbb{Z}} |a_n v_i^{n+1}| \leq \sum_{n\leq -1} C\left(\frac{\varepsilon^2}{2t^2}\right)^{n+1} + \sum_{n>-1} C(t/\varepsilon)^{n+1} \leq C'$$

$$\sum_{n\in\mathbb{Z}} |t^2 a_n v_i^{n-1}| \leq \sum_{n\leq 1} C\varepsilon^2\left(\frac{\varepsilon^2}{2t^2}\right)^{n-1} + \sum_{n>1} C(t/\varepsilon)^{n+1} \leq C'$$

for some constant C'. Hence

$$\int_{v_i=t}^{v_i=\varepsilon/2} \sum_{n\neq -1} a_n v_i^{n+1} dv_i \quad \text{and} \quad \int_{v_i=t}^{v_i=\varepsilon/2} \sum_{n\neq 1} t^2 a_n v_i^{n-1} dv_i$$

extend analytically to $t = 0$. (Proof: again think of t as a complex parameter and use the Riemann Extension Theorem.) Not so for the terms $a_{-2}\log(\varepsilon/2t)$ and $t^2 a_0 \log(\varepsilon/2t)$ because these are multi-valued functions of t, seen as a complex number, so Riemann Extension Theorem cannot be used. But using (8), we see that $t^{-2}a_{-2}$ is bounded, so extends analytically to $t = 0$. So the log terms are of the form $t^2 \log t \times$ analytic. This proves Lemma 2. q.e.d.

Proof of Proposition 13. Assume that α, β are given by Proposition 11, r is given by Proposition 12 and $t = \exp(-1/\tau^2)$. Define

$$\mathcal{F}_i(\tau, a, b, \mathrm{R}) = tP(\Gamma_i), \qquad \mathcal{F} = (\mathcal{F}_i)_{i\in J}.$$

By the lemma, $\mathcal{F}_i$ is a smooth function and

$$\mathcal{F}_i(0, a, b, \mathrm{R}) = b_{i_0} + \overline{a_{i_0}} - b_i - \overline{a_i}$$

up to sign and conjugation, depending on whether ℓ_i is odd or even. Hence $\mathcal{F}(0, a^0, b^0, \mathrm{R}^0) = 0$. We normalise the b parameters by fixing

$$\forall k, \quad 1 \leq k \leq N-1, \quad b_{i_0(k)} = -\overline{a_{i_0(k)}}.$$

This normalises translation in $\mathbb{C}_k$ (see Remark 4). The partial differential of $\mathcal{F}$ with respect to the remaining b parameters is an ($\mathbb{R}$-linear) isomorphism from $\mathbb{C}^{n-N+1}$ to $\mathbb{C}^{n-N+1}$. By the Implicit Function Theorem, for (τ, a, R) in a neighborhood of $(0, a^0, \mathrm{R}^0)$, there exists a unique b, satisfying the above normalisation, such that $\mathcal{F}(\tau, a, b, \mathrm{R}) = 0$. The last statement of the proposition follows from uniqueness and normalisation. q.e.d.

3.8 Horizontal γ-periods

The last equations we have to solve are

$$\forall i \in J, \quad P(\gamma_i) = 0 \quad \text{and} \quad \forall k, 1 \leq k \leq N, \quad P(\delta_k) = 0$$

where δ_k is a small circle around ∞_k. Using the Residue Theorem, these equation are equivalent to

$$\forall i \in I, \quad P(\gamma_i) = 0.$$

Proposition 14. *Assume that α, β are given by Proposition 11, r is given by Proposition 12, b is given by Proposition 13 and $t = \exp(-1/\tau^2)$. For τ in a neighborhood of 0, there exists $a = (a_1, \dots, a_n)$ and $\mathrm{R} = (\mathrm{R}_1, \dots, \mathrm{R}_N)$, depending smoothly on τ, such that $\mathrm{R}_1 + \cdots + \mathrm{R}_N = 0$ and the corresponding Weierstrass data satisfies $P(\gamma_i) = 0$ for all $i \in I$. Moreover, when $\tau = 0$, we have $a = a^0$ and $\mathrm{R} = \mathrm{R}^0$.*

We need:

Lemma 3. *With the same hypotheses, consider some k, $1 \leq k \leq N-1$, and some $i \in I_k$. If $\tau \neq 0$, let*

$$\mathcal{F}_i(\tau, a, \mathrm{R}) = \frac{1}{t} P(\gamma_i).$$

Then $\mathcal{F}_i$ extends to a smooth function at $\tau = 0$. Moreover,

$$\begin{aligned} &\mathcal{F}_i(0, a, \mathrm{R}) \\ &= 4\pi \mathrm{i}\ (-1)^{k+1} \left(\sum_{j \in I_k,\, j \neq i} \frac{2c_k^2}{\overline{p_i} - \overline{p_j}} - \sum_{j \in I_{k-1}} \frac{c_k c_{k-1}}{\overline{p_i} - \overline{p_j}} - \sum_{j \in I_{k+1}} \frac{c_k c_{k+1}}{\overline{p_i} - \overline{p_j}} \right) \end{aligned}$$

where c_k is defined in function of R in Proposition 12 and p_i is defined in function of a_i by

$$p_i = \begin{cases} \overline{a_i} & \text{if } i \in I_k,\ k \text{ odd} \\ -a_i & \text{if } i \in I_k,\ k \text{ even.} \end{cases}$$

Proof. First assume that k is odd, so $g = t g_k$ in $\mathbb{C}_k$ and $g = (t g_{k+1})^{-1}$ in $\mathbb{C}_{k+1}$. Recall that γ_i is homologous to the circle $|z - a_i| = \varepsilon$ in $\mathbb{C}_k$, with the negative orientation, and to the circle $|z - b_i| = \varepsilon$ in $\mathbb{C}_{k+1}$, with the positive orientation. Hence

$$\frac{1}{t} \int_{\gamma_i} g\, dh = -\int_{|z - a_i| = \varepsilon} g_k dh$$

$$\frac{1}{t} \int_{\gamma_i} g^{-1} dh = \int_{|z - b_i| = \varepsilon} g_{k+1} dh.$$

Since the circles are contained in Ω, the right terms are analytic functions of all parameters (including at $t = 0$), hence smooth functions of (τ, a, R) when $t, \alpha, \beta, \mathrm{r}$ are as in the proposition. When $\tau = 0$, we have

$dh = g_k dz$ on $\mathbb{C}_k$ so

$$\begin{aligned}\int_{|z-a_i|=\varepsilon} g_k dh &= 2\pi \mathrm{i}\ \mathrm{Res}_{a_i} g_k^2 \\ &= 2\pi\mathrm{i}\ \mathrm{Res}_{a_i}\left(\sum_{j\in I_k}\frac{-c_k}{z-a_j}+\sum_{j\in I_{k-1}}\frac{c_{k-1}}{z-b_j}\right)^2 \\ &= 4\pi\mathrm{i}\left(\sum_{j\in I_k\ ,\ j\neq i}\frac{c_k^2}{a_i-a_j}-\sum_{j\in I_{k-1}}\frac{c_k c_{k-1}}{a_i-b_j}\right)\\ \int_{|z-b_i|=\varepsilon} g_{k+1} dh &= 4\pi\mathrm{i}\left(\sum_{j\in I_k\ ,\ j\neq i}\frac{c_k^2}{b_i-b_j}-\sum_{j\in I_{k+1}}\frac{c_k c_{k+1}}{b_i-a_j}\right).\end{aligned}$$

Hence using that $b_j = -\overline{a_j}$ we obtain

$$\mathcal{F}_i(0,a,\mathrm{R}) = 4\pi\mathrm{i}\left(2\sum_{j\in I_k,\, j\neq i}\frac{c_k^2}{a_i-a_j}-\sum_{j\in I_{k+1}}\frac{c_k c_{k+1}}{a_i+\overline{a_j}}-\sum_{j\in I_{k-1}}\frac{c_k c_{k-1}}{a_i+\overline{a_j}}\right).$$

When k is even the computation is similar:

$$\frac{1}{t}\int_{\gamma_i} g\ dh = \int_{|z-b_i|=\varepsilon} g_{k+1} dh$$

$$\frac{1}{t}\int_{\gamma_i} g^{-1} dh = -\int_{|z-a_i|=\varepsilon} g_k dh$$

$$\mathcal{F}_i(0,a,\mathrm{R}) = 4\pi\mathrm{i}\left(2\sum_{j\in I_k,\, j\neq i}\frac{c_k^2}{\overline{a_i}-\overline{a_j}}-\sum_{j\in I_{k+1}}\frac{c_k c_{k+1}}{\overline{a_i}+a_j}-\sum_{j\in I_{k-1}}\frac{c_k c_{k-1}}{\overline{a_i}+a_j}\right).$$

This gives the formula of Lemma 3. q.e.d.

Proof of Proposition 14. Observe that

$$\mathcal{F}_i(0,a,\mathrm{R}) = 4\pi\mathrm{i}\,(-1)^{k+1}\overline{F(p)}$$

where F is the force defined in the introduction. When $a = a^0$ and $\mathrm{R} = \mathrm{R}^0$ we have $p = p^0$ and $c = c^0$, hence since the configuration p^0 is balanced, $\mathcal{F}_i(0,a^0,\mathrm{R}^0) = 0$. Fix some indices i_1, i_2 and j_1, j_2 such that $p_{i_1} \neq p_{i_2}$ and $p_{j_1} \neq p_{j_2}$. By non-degeneracy (see Remark 6 below), the partial differential of $(\mathcal{F}_i)_{i\neq i_1\ ,\ i\neq i_2}$ with respect to the variables

$(a_j)_{j\neq j_1\,,\,j\neq j_2}$ is an isomorphism from $\mathbb{C}^{n-2}$ to $\mathbb{C}^{n-2}$ (this operator is only $\mathbb{R}$-linear because of the conjugations).

We normalise the a parameters by fixing $a_{j_1}=a^0_{j_1}$, $a_{j_2}=a^0_{j_2}$. By the Implicit Function Theorem, for (τ,R) in a neighborhood of $(0,\mathrm{R}^0)$, there exists $n-2$ smooth functions a_j, $j\neq j_1$, $j\neq j_2$, such that

$$\forall i\ ,\ i\neq i_1\ ,\ i\neq i_2,\quad \mathcal{F}_i(\tau,a,\mathrm{R})=0.$$

Remark 6. Let A be the complex matrix $(\partial F_i/\partial p_j)_{1\leq i,j\leq n}$. Non-degenerate means that the matrix A has an invertible minor of size $n-2$. However, from Equations (1) and (2), A satisfies

$$\forall j,\quad \sum_i A_{i,j}=0,\qquad \sum_i p_i\,A_{i,j}=0.$$

Note also that A is symmetric. This implies that for any i_1, i_2 and j_1, j_2 such that $p_{i_1}\neq p_{i_2}$ and $p_{j_1}\neq p_{j_2}$, the minor obtained by removing the rows i_1, i_2 and the columns j_1, j_2, is invertible.

It remains to prove that $\mathcal{F}_{i_1}=\mathcal{F}_{i_2}=0$. We do this as follows. First we find two equations (one complex, one real) satisfied by the periods. Then we use the parameters R_k to obtain one more real relation.

We choose the indices i_1 and i_2 as follows. It is easy to see that if $n_k=1$ for all k, the configuration cannot be balanced, unless $n=1$, which we excluded. Hence there exists k_0 such that $n_{k_0}\geq 2$. Let $i_1=\min I_{k_0}$ and choose $i_2\in I_{k_0}$, $i_2>i_1$.

Lemma 4. *Assume that all parameters are as above. Then*

$$P(\gamma_{i_1})+P(\gamma_{i_2})=0$$

$$\mathrm{Re}\left(P(\gamma_{i_2})\int_{\Gamma_{i_2}} g^{-1}dh\right)=0.$$

Proof. Note that $g\,dh$ and $g^{-1}dh$ only have poles at ∞_k. By the residue Theorem,

$$\sum_{i\in I_{k-1}}P(\gamma_i)-\sum_{i\in I_k}P(\gamma_i)=\begin{cases}\overline{2\pi\mathrm{i}\ \mathrm{Res}_{\,\infty_k}g^{-1}dh} & \text{if } k \text{ odd}\\ -2\pi\mathrm{i}\ \mathrm{Res}_{\,\infty_k}g\ dh & \text{if } k \text{ even.}\end{cases}$$

The left side is zero unless $k=k_0$ or $k=k_0+1$. Hence at most one residue of $g\,dh$ is nonzero and at most one residue of $g^{-1}dh$ is nonzero.

Since the sum of the residues is zero, all residues of $g^{\pm 1}dh$ are zero. This implies the first statement.

The second statement comes from the Riemann Bilinear Relation for a pair of meromorphic differentials ω, ω' whose poles have no residues (see [3] page 241)

$$\sum_{i=1}^{G}\int_{\gamma_i}\omega\int_{\Gamma_i}\omega' - \int_{\gamma_i}\omega'\int_{\Gamma_i}\omega = 2\pi\mathrm{i}\sum \mathrm{Res}\,(f\omega')$$

where $df = \omega$ (f is well defined in a neighborhood of each pole). We use this formula with $\omega = g\,dh$ and $\omega' = g^{-1}dh$. We first compute the residues. Assume that g has a simple zero at ∞_k. Use $w = g$ as a local coordinate in a neighborhood of ∞. Since the residue of $g^{-1}dh$ is zero,

$$dh = (\mathrm{R}_k w^{-1} + \mathcal{O}(w))dw$$

$$f\omega' = (\mathrm{R}_k w + O(w^3))(\mathrm{R}_k w^{-2} + \mathcal{O}(1))dw$$

$$\mathrm{Res}\, f\omega' = \mathrm{R}_k^2.$$

Similar computations give the same result with a minus sign when g has a simple pole, and zero when g has a multiple zero or pole. The important point for us is that the residue is real. Using that

$$\int_{\Gamma_i} g\ dh = \overline{\int_{\Gamma_i} g^{-1}dh}$$

$$\int_{\gamma_i} g\ dh = \overline{\int_{\gamma_i} g^{-1}dh} - P(\gamma_i)$$

we obtain from Riemann Bilinear Relation

$$\sum_{i\in J}\left(2\mathrm{i}\ \mathrm{Im}\left(\overline{\int_{\gamma_i} g^{-1}dh}\int_{\Gamma_i} g^{-1}dh\right) - P(\gamma_i)\int_{\Gamma_i} g^{-1}dh\right) = 2\pi\mathrm{i}\sum_{k=1}^{N}\pm\mathrm{R}_k^2.$$

Taking the real part gives the second statement of Lemma 4. q.e.d.

End of proof of Proposition 14.

Assume all parameters are as above, so that $\mathcal{F}_i = 0$ if $i \neq i_1, i_2$. Let

$$\mathcal{G}(\tau,\mathrm{R}) = \mathrm{Im}\left(\sum_k\sum_{i\in I_k}(-1)^{k+1}\overline{p_i}\mathcal{F}_i\right).$$

This is a smooth function with

$$\mathcal{G}(0, \mathrm{R}) = \mathrm{Im}\left(4\pi\mathrm{i}\sum_{i\in I}\overline{p_i}\overline{F_i}\right) = 4\pi\mathcal{W}.$$

By hypothesis, the differential of $\mathcal{W}$ with respect to $(c_1, \ldots, c_{N-1})$ is surjective. Note that $(c_1, \ldots, c_{N-1}) \mapsto (\mathrm{R}_1, \ldots, \mathrm{R}_N)$ is an isomorphism from $\mathbb{R}^{N-1}$ to $\{\mathrm{R} \in \mathbb{R}^N \mid \mathrm{R}_1 + \cdots + \mathrm{R}_N = 0\}$. Hence the partial differential of $\mathcal{G}$ with respect to R is surjective. By the Implicit Function Theorem, using a supplementary space of the kernel, for τ in a neighborhood of 0, there exists $\mathrm{R} = (\mathrm{R}_1, \ldots, \mathrm{R}_N)$ such that $\mathrm{R}_1 + \cdots + \mathrm{R}_N = 0$ and $\mathcal{G}(\tau, \mathrm{R}) = 0$.

Let us now conclude. From $\mathcal{G} = 0$ and the first statement of Lemma 4, we have

$$\mathrm{Im}((\overline{p_{i_2}} - \overline{p_{i_1}})\mathcal{F}_{i_2}) = 0.$$

From the second statement of Lemma 4, we have

$$\mathrm{Re}(\lambda\mathcal{F}_{i_2}) = 0, \quad \text{with} \quad \lambda = t\int_{\Gamma_{i_2}} g^{-1}dh \simeq (-1)^{k_0}(\overline{p_{i_2}} - \overline{p_{i_1}}).$$

Hence $\mathcal{F}_{i_2} = 0$ and $\mathcal{F}_{i_1} = 0$. This concludes the proof of Proposition 14.
q.e.d.

Remark 7. The kernel of $\partial\mathcal{G}/\partial\mathrm{R}$ has dimension $N - 2$, so we have $N - 2$ free parameters amongst the logarithmic growths. Together with the parameter t, this gives a family depending on $N - 1$ real parameters, the expected dimension for the space of embedded minimal surfaces with N ends modulo translation and rotation.

3.9 Geometry of M_t

What we have achieved so far is the following. For $t > 0$ close to 0 we have found values of the parameters $(\alpha, \beta, a, b, \mathrm{r}, \mathrm{R})$ depending smoothly on $t > 0$ such that the corresponding Weierstrass data satisfies the zero/pole condition and has no periods. When $t \to 0$, the parameters converge to the value $(\alpha^0, \beta^0, a^0, b^0, \mathrm{r}^0, \mathrm{R}^0)$ given in Section 3.3, but they are not smooth functions of t at $t = 0$ (they only depend smoothly on $\tau = 1/\sqrt{|\log t|}$). What remains to be proven is that the minimal surface M_t given by this Weierstrass data satisfies the conclusions of Theorem 1 and in particular prove embeddedness.

Let 0_k be the point $z = 0$ in $\mathbb{C}_k$ (we may assume by translation that $0_k \in \Omega_k$). The Weierstrass formula, starting integration at $z_0 = 0_1$, defines a minimal immersion $X = (X_1, X_2, X_3) : \Sigma \to \mathbb{R}^3$ with embedded ends at $\infty_1, \dots, \infty_N$. If $\mathrm{R}_k \neq 0$, then the end at ∞_k is catenoidal with logarithmic growth R_k, while if $\mathrm{R}_k = 0$, it is asymptotic to a plane. Let

$$T_i = (T_{i,1}, T_{i,2}, T_{i,3}) = \frac{1}{2} \left(X(v_i = t) + X(v_i = -t) \right).$$

Proposition 15.

1. *When $t \to 0$, we have, for $1 \leq k \leq N$,*

$$(X_1 + \mathrm{i}\, X_2)(0_k) = o(t^{-1}),$$
$$X_3(0_k) = 2|\log t|(c_1 + \cdots + c_{k-1}) + o(\log t).$$

   For each $i \in I_k$ we have

$$T_{i,1} + \mathrm{i}\, T_{i,2} = \frac{p_i}{2t} + o(t^{-1}),$$
$$T_{i,3} = \frac{1}{2} \left(X_3(0_k) + X_3(0_{k+1}) \right) + o(\log t).$$

   Hence up to scaling by $2t$, p_i is the limit position of the i^{th} neck when $t \to 0$.

2. *Given $\delta > 1$, for each $i \in I_k$, the image of the domain $t/\delta < |v_i| < \delta t$, translated by $-T_i$, converges when $t \to 0$ to the catenoid with waist radius c_k, with center at the origin, intersected with the slab $|x_3| < c_k \log \delta$.*

3. *For each k, the image of the domain of $\mathbb{C}_k$ defined by $|v_i| > t$, $i \in I_k$ and $|w_i| > t$, $i \in I_{k-1}$, is a graph over a domain in the horizontal plane. Moreover, for any $\epsilon > 0$ and $\delta > 1$:*

   3a. *The image of the domain of $\mathbb{C}_k$ defined by $|v_i| > \epsilon$, $i \in I_k$ and $|w_i| > \epsilon$, $i \in I_{k-1}$, stays at bounded distance from the graph*

$$x_3 = X_3(0_k) + \mathrm{R}_k \log(1 + t|x_1 + \mathrm{i}\, x_2|).$$

   3b. *For $i \in I_k$, the image of the domain $\delta t < |v_i| < \epsilon$ is inside the cylinder with vertical axis passing through T_i and radius $C\epsilon/t$ (where C is a constant independent of ϵ and t), intersected* *with the slab $X_3(0_k) < x_3 < T_{i,3} - c_k \log\delta$. The image of the domain $\delta t < |w_i| < \epsilon$ is inside the same cylinder, intersected with the slab $T_{i,3} + c_k \log\delta < x_3 < X_3(0_{k+1})$.*

4. *If $Q_1 < Q_2 < \cdots < Q_N$, then M_t is embedded for t small enough.*
5. *If $Q_k \neq 0$, let μ_k be the intersection of the axis of the catenoidal end ∞_k with the horizontal plane. Then*

$$\mu_k = \frac{1}{2t}\left(\sum_{i\in I} Q_{i,k}\, p_i\right) \Big/ \left(\sum_{i\in I} Q_{i,k}\right) + o(t^{-1}).$$

Note that we do not write the upper-scripts 0 in this proposition, so for instance c_k and p_i are the size and position of the necks as given in the configuration. Recall that for $i \in I_k$, the limit of r_i when $t \to 0$ is c_k, and the limit of R_k is Q_k.

Proof of 1. By the computation of Section 3.6,

$$X_3(0_{k+1}) - X_3(0_k) = \mathrm{Re}\int_{0_k}^{0_{k+1}} dh = -\mathrm{r}_i \log t^2 + \text{bounded}$$

where $i \in I_k$ (the result does not depend on i precisely because the real period of dh along Γ_i is zero).

$$X_3(v_i = \pm t) - X_3(0_k) = \mathrm{Re}\int_{z=0_k}^{v_i=\pm t} dh = -\mathrm{r}_i \log t + \text{bounded}.$$

By formula (9), if k is, say, odd,

$$\begin{aligned}&(X_1 + \mathrm{i}\, X_2)(0_{k+1}) - (X_1 + \mathrm{i}\, X_2)(0_k)\\ &= \frac{1}{2t}\left(\int_0^{a_i} \overline{dz} + \int_{b_i}^{0} -dz\right) + o(t^{-1}) = o(t^{-1})\end{aligned}$$

$$\begin{aligned}&(X_1 + \mathrm{i}\, X_2)(v_i = \pm t) - (X_1 + \mathrm{i}\, X_2)(0_k)\\ &= \frac{1}{2t}\int_0^{a_i} \overline{dz} + o(t^{-1}) = \frac{p_i}{2t} + o(t^{-1}).\end{aligned}$$

Proof of 2. Let $u = v_i/t$ so $1/\delta < |u| < \delta$ in this domain. Using the notations of Section 3.6 we have,

$$g = (t/v_i)^{(-1)^{k+1}} = u^{(-1)^k}$$

$$dh = \sum_{n\in\mathbb{Z}} a_n v_i^n \, dv_i = \sum_{n\in\mathbb{Z}} a_n t^{n+1} u^n \, du.$$

The estimate (8) imply that

$$|a_n t^{n+1}| \leq C(\varepsilon/t)^{n+1} \to 0 \text{ if } n \leq -2$$

$$|a_n t^{n+1}| \leq C(t/\varepsilon)^{n+1} \to 0 \text{ if } n \geq 0$$

$$a_{-1} = -r_i \to -c_k.$$

So the Weierstrass data converges when $t \to 0$ to the Weierstrass data of a catenoid $g = u^{\pm 1}$, $dh = -c_k du/u$.

Proof of 3. By definition of the Gauss map we have, in this domain, $|g| < 1$ if k is odd and $|g| > 1$ if k is even, so the normal stays in either the lower or upper hemisphere. Hence the projection π on the horizontal plane is a local diffeomorphism. From its behaviour in a neighborhood of the boundary circles and at infinity, we conclude that π is a diffeomorphism onto its image. (Proof. This is a topological issue. From the convergence to catenoids, π maps homeomorphically the circle $|v_i| = t$ to a circle in the plane, so we may extend π to a local homeomorphism $\widetilde{\pi} : \mathbb{C}_k \to \mathbb{C}$. From the behaviour at the ends, $\widetilde{\pi} : \mathbb{C}_k \cup \{\infty_k\} \to \mathbb{C} \cup \{\infty\}$ is a local homeomorphism. Compactness at the source plus local homeomorphism implies that $\widetilde{\pi}$ is a covering map. In a neighborhood of ∞_k there is only one sheet so $\widetilde{\pi}$ is a homeomorphism, hence π is a diffeomorphism onto its image.)

Proof of 3a. By (9) we have, for $|z| \gg 1$,

$$|(X_1 + \mathrm{i}\, X_2)(z)| = \frac{|z|}{2t} + o(t^{-1}).$$

Since dh has a simple pole at ∞_k with residue $-\mathrm{R}_k$,

$$\begin{aligned} X_3(z) - X_3(0_k) &= \mathrm{R}_k \log(1 + |z|) + \text{bounded} \\ &= \mathrm{R}_k \log(1 + t|X_1 + \mathrm{i}\, X_2|) + \text{bounded}. \end{aligned}$$

The estimate 3b is a consequence of (8).

Proof of 4. If $Q_k < Q_{k+1}$ then $R_k \leq R_{k+1}$ for t small enough, hence by 3a and 3b, the domains of point 3, for varying k, are disjoint. This implies that M_t is embedded. (A formal proof may be written as follows: M_t may be covered by suitable open sets of $\mathbb{R}^3$ so that the intersection of M_t with each set is included either in a domain of point 2 or point 3, so is embedded.)

Proof of 5. By definition (see [5], Section 2.3.2), μ_k is the unique point in the horizontal plane such that $\text{Torque}(\mu_k, \delta_k) = 0$, where δ_k is a small circle around the end ∞_k, and Torque is the homology-invariant vector

$$\text{Torque}(X_0, \gamma) = \int_\gamma (X - X_0) \wedge \nu$$

where ν is the exterior conormal. Using the homology invariance in the domain Ω_k, this is equivalent to

$$\sum_{i \in I_k} \text{Torque}(\mu_k, \gamma_i) - \sum_{i \in I_{k-1}} \text{Torque}(\mu_k, \gamma_i) = 0.$$

We have the straightforward formula

$$\text{Torque}(\mu_k, \gamma_i) = \text{Torque}(T_i, \gamma_i) + (T_i - \mu_k) \wedge \text{Flux}(\gamma_i).$$

From the convergence to a catenoid with axis passing at T_i we obtain

$$\lim_{t \to 0} \text{Torque}(T_i, \gamma_i) = 0$$

$$\lim_{t \to 0} \text{Flux}(\gamma_i) = (0, 0, 2\pi c_k).$$

Looking at the horizontal part of the Torque, identifying $\mathbb{R}^2$ with $\mathbb{C}$, we obtain

$$-2\pi i \left(\sum_{i \in I_k} c_k \left(\frac{p_i}{2t} - \mu_k \right) - \sum_{i \in I_{k-1}} c_{k-1} \left(\frac{p_i}{2t} - \mu_k \right) \right) = o(t^{-1}).$$

Recalling the definition of the charges $Q_{i,k}$ we obtain the result. This concludes the proof of the proposition and Theorem 1. q.e.d.

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